



On superlinear problems without the Ambrosetti and Rabinowitz condition[☆]

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ABSTRACT

Existence and multiplicity results are obtained for superlinear p -Laplacian equations without the Ambrosetti and Rabinowitz condition. To overcome the difficulty that the Palais–Smale sequences of the Euler–Lagrange functional may be unbounded, we consider the Cerami sequences. Our results extend the recent results of Miyagaki and Souto [O. Miyagaki, M. Souto, Superlinear problems without Ambrosetti and Rabinowitz growth condition, *J. Differential Equations* 245 (2008) 3628–3638].

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1. Introduction

Let Ω be a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$. We consider the following Dirichlet problem for the p -Laplacian equation ($p > 1$):

$$\begin{cases} -\Delta_p u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator, and the nonlinearity $f(x, u)$ satisfies the following conditions:

(f_*) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exist $C > 0$ and $q \in (p, p^*)$ such that for $(x, t) \in \Omega \times \mathbb{R}$ we have

$$|f(x, t)| \leq C(1 + |t|^{q-1});$$

where $p^* = Np/(N - p)$ if $N > p$, and $p^* = \infty$ if $N \leq p$.

(f_1) The following limit holds uniformly for a.e. $x \in \Omega$:

$$\lim_{|t| \rightarrow \infty} \frac{F(x, t)}{|t|^p} = +\infty, \quad \text{where } F(x, t) = \int_0^t f(x, s) ds. \quad (1.2)$$

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(f₂) There exists $R > 0$ such that for $\forall x \in \Omega$,

$$\frac{f(x, t)}{|t|^{p-2} t} \text{ is increasing in } t \geq R, \text{ and decreasing in } t \leq -R.$$

Note that (f₂) is a condition for $|t| \geq R$; therefore it is a quite generic assumption.

The condition (f₁) is a consequence of the following condition:

$$\lim_{|t| \rightarrow \infty} \frac{f(x, t)}{|t|^{p-2} t} = +\infty. \tag{1.3}$$

In the case $p = 2$ this condition characterizes the problem (1.1) as superlinear at infinity. Since the seminal work of Ambrosetti and Rabinowitz [1], such superlinear elliptic boundary value problems have been studied by many authors; see for instance [2–7]. A common feature of these works is that the following condition, which was originally due to Ambrosetti and Rabinowitz [1] for the case $p = 2$, is imposed on the nonlinearity $f(x, t)$.

(AR) There exist $\mu > p$ and $R > 0$ such that

$$0 < \mu F(x, t) \leq f(x, t)t, \quad \text{for } x \in \Omega \text{ and } |t| \geq R.$$

The role of (AR) is to ensure the boundedness of the Palais–Smale sequences of the Euler–Lagrange functional Φ given by (2.2). This is very much crucial in the applications of critical point theory. However, although (AR) is a quite natural condition, it is somewhat restrictive and eliminates many nonlinearities. In fact, (AR) implies that for some $c, d > 0$,

$$F(x, t) \geq c |t|^\mu - d. \tag{1.4}$$

Hence, for example, the function

$$f(x, t) = |t|^{p-2} t \log(1 + |t|)$$

does not satisfy (AR) for any $\mu > p$. But it satisfies our conditions (f_{*}), (f₁) and (f₂).

For this reason, in recent years, some authors [8–15] studied the superlinear problem (1.1) trying to drop the condition (AR). We refer the readers to [12, Section 1] for a detailed discussion on the assumptions of [8,13,14].

Assume that $f(x, 0) = 0$; then the zero function $u = \mathbf{0}$ is a trivial solution of (1.1). Therefore we are interested in the existence of nontrivial solutions.

Theorem 1.1. Assume that (f_{*}), (f₁), (f₂) and

$$(f_0^1) \quad f(x, 0) = 0, f(x, t) = o(|t|^{p-2} t) \text{ as } |t| \rightarrow 0 \text{ uniformly in } x \in \Omega.$$

Then the problem (1.1) has a nontrivial (weak) solution in $W_0^{1,p}(\Omega)$.

For the case $p = 2$ this result is a finding due to Miyagaki and Souto [12]. Without assuming (AR) the Euler–Lagrange functional Φ may possess unbounded Palais–Smale sequences. To overcome this difficulty, Miyagaki and Souto considered a family of perturbed problems

$$-\Delta u = \lambda f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{1.5}$$

and adapted some monotonicity arguments used by Struwe and Tarantello [16] and Schechter and Zou [13]. This approach is interesting, but the disadvantage is also obvious: many powerful variational tools such as Morse theory and the Fountain Theorem are not directly applicable.

In [10], Liu and Wang studied (1.1) the case $p = 2$ under the assumptions (f_{*}), (f₁), (f₀¹), and

$$(f_4) \quad f(x, t)t - 2F(x, t) \text{ is nondecreasing in } |t| \text{ and increasing for } |t| \text{ small.}$$

Note that (f₄) is a global condition. Using a Nehari type argument, they obtained three solutions, one of which is sign-changing. Such a Nehari type argument is also used in [17] to produce ground states of superlinear Schrödinger equations, whose nonlinearity does not satisfy (AR).

For the case $p = 2$, our approach can also provide three solutions (see Remark 3.1). Combining our approach and the variational technique in partially ordered Hilbert space (see [18] and references therein), it is also possible to discuss the nodal property of the solutions that we obtain.

We will prove Theorem 1.1 for general $p > 1$ using an approach different to that of [12,10]. The key point in our proof is that, although Φ may possess unbounded Palais–Smale sequences, under the assumptions of the theorem, the functional Φ satisfies the Cerami condition. This is just as in our previous work [19,9], where the superlinear problem (1.1) is studied under the following condition introduced by Jeanjean [11] for the case $p = 2$:

$$(je) \quad \text{there exists } \theta \geq 1 \text{ such that } \theta \mathcal{F}(x, t) \geq \mathcal{F}(x, st) \text{ for } (x, t) \in \Omega \times \mathbb{R} \text{ and } s \in [0, 1], \text{ where } \mathcal{F}(x, t) = f(x, t)t - pF(x, t).$$

Although (Je) is weaker than the assumption that

$$\frac{f(x, t)}{|t|^{p-2}t} \text{ is increasing in } t \geq 0, \text{ and decreasing in } t \leq 0, \quad (1.6)$$

both (Je) and (1.6) are *global* conditions on $f(x, t)$, and therefore are not very satisfactory. In Lemma 2.5 we will show that under the *local* condition (f_2) near infinity, the functional Φ satisfies the Cerami condition. Then Theorem 1.1 follows directly from the classical Mountain Pass Theorem.

Since Φ satisfies the Cerami condition, $C_*(\Phi, \infty)$, the critical groups at infinity, make sense. This enables us to compute $C_*(\Phi, \infty)$ and apply Morse theory to consider the following situation, which cannot be dealt with via the approach of Miyagaki and Souto [12].

Let λ_1 and λ_2 be the first and the second eigenvalues of $-\Delta_p$ on $W_0^{1,p}(\Omega)$. It is well known that $\lambda_1 > 0$ is a simple eigenvalue, and that $\sigma(-\Delta_p) \cap (\lambda_1, \lambda_2) = \emptyset$, where $\sigma(-\Delta_p)$ is the spectrum of $-\Delta_p$, (cf. [20]). We assume the following condition on the nonlinearity near zero.

(f_0^2) There exist $\rho > 0$ and $\bar{\lambda} \in (\lambda_1, \lambda_2)$ such that

$$\lambda_1|t|^p \leq pF(x, t) \leq \bar{\lambda}|t|^p, \quad \text{for } x \in \Omega \text{ and } |t| \leq \rho.$$

Then we have the following theorem.

Theorem 1.2. Assume that (f_*) , (f_0^2) , (f_1) and (f_2) are satisfied; then the problem (1.1) has a nontrivial solution in $W_0^{1,p}(\Omega)$.

Remark 1.3. (i) For the case $p \neq 2$, in the literature, for finding *nontrivial* solutions most attention is paid to the case where $\mathbf{0}$ is a local minimizer of Φ , and then the Mountain Pass Theorem is applied. However, under our assumption (f_0^2) , $\mathbf{0}$ is not a local minimizer of Φ . This case has been considered in [5,9], where a nontrivial solution is obtained under the condition (f_*) , (f_0^2) , as well as (AR), or (Je) and (1.3).

(ii) In (f_0^2) only the first two eigenvalues are involved. As in [6], on the basis of our computation of $C_*(\Phi, \infty)$, we may consider some situations involving higher eigenvalues. Assume that

$$\lambda = \lim_{|t| \rightarrow 0} \frac{f(x, t)}{|t|^{p-2}t} \notin \sigma(-\Delta_p).$$

If (f_*) , (f_1) and (f_2) are satisfied, then (1.1) has a nontrivial solution; see the proof of [9, Theorem 1.3].

Another advantage of the fact that Φ satisfies the Cerami condition is that we can easily obtain multiplicity results in the case where $f(x, t)$ is odd in t .

Theorem 1.4. Assume that (f_*) , (f_1) , (f_2) and

(f) there exists $\Lambda > 0$ such that $F(x, t) \geq -\Lambda|t|^p$ for $(x, t) \in \Omega \times \mathbb{R}$

are satisfied. If $f(x, t) = -f(x, -t)$, then (1.1) has a sequence of solutions $\{u_n\}$ such that $\Phi(u_n) \rightarrow +\infty$.

Remark 1.5. For related results on the existence of infinitely many solutions of superlinear elliptic problems, we refer the reader to [15,19,10]. In [15], the growth condition (1.4) is required, while in [19], the global condition (Je) is assumed. In [10, Theorem 2.3], in addition to the global condition (f_4) , it is also essential to require that f is of class C^1 .

The paper is organized as follows: as preliminaries, in Section 2 we discuss the Cerami condition and compute the relevant critical groups of the functional Φ ; in Section 3 we prove our theorems.

2. The Cerami condition and critical groups

Let Φ be a C^1 -functional defined on a Banach space X ; then the k -th critical group of Φ at an isolated critical point u with $\Phi(u) = c$ is defined by

$$C_k(\Phi, u) := H_k(\Phi_c, \Phi_c \setminus \{u\}), \quad k \in \mathbb{N} = \{0, 1, 2, \dots\},$$

where H_* is the singular relative homology with coefficients in an Abelian group \mathcal{G} and $\Phi_c = \Phi^{-1}(-\infty, c]$.

We say that Φ satisfies the Cerami condition (C) if any sequence $\{u_n\} \subset X$ such that $\{\Phi(u_n)\}$ is bounded and $(1 + \|u_n\|) \|\Phi'(u_n)\| \rightarrow 0$ has a convergent subsequence; such a sequence is then called a Cerami sequence. If Φ satisfies the condition (C) and the critical values of Φ are bounded from below by some $\alpha > -\infty$, then the critical groups of Φ at infinity are those introduced by Bartsch and Li [21] as

$$C_k(\Phi, \infty) := H_k(X, \Phi_\alpha), \quad k \in \mathbb{N}. \quad (2.1)$$

Note that by the deformation lemma, the right-hand side of (2.1) does not depend on the choice of α .

The reader is referred to [22,23] for more details on Morse theory. In the proofs of Theorem 1.2 we shall use the following result.

Proposition 2.1. *Suppose that $\Phi \in C^1(X, \mathbb{R})$ satisfies the condition (C) and Φ has only finitely many critical points. Let $\mathbf{0}$ be an isolated critical point of Φ . If for some $k \in \mathbb{N}$ we have $C_k(\Phi, \mathbf{0}) \neq C_k(\Phi, \infty)$, then Φ has a nonzero critical point.*

Remark 2.2. The result of Proposition 2.1 is well known under the (PS) condition. It is also true under the condition (C) because in this situation the second deformation lemma is still valid; see [24, Lemma 4.2].

The condition (f_*) implies that the functional $\Phi : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$:

$$\Phi(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} F(x, u) \, dx \tag{2.2}$$

is well defined and of class C^1 . It is well known that the critical points of Φ are weak solutions of (1.1). In order to find critical points of Φ , we must show that Φ satisfies the condition (C).

Lemma 2.3. *If (f_2) holds, then for any $x \in \Omega$, $\mathcal{F}(x, t)$ is increasing in $t \geq R$ and decreasing in $t \leq -R$, where $\mathcal{F}(x, t) = f(x, t)t - pF(x, t)$. In particular, there exists $C_1 > 0$ such that*

$$\mathcal{F}(x, s) \leq \mathcal{F}(x, t) + C_1 \tag{2.3}$$

for $x \in \Omega$ and $0 \leq s \leq t$ or $t \leq s \leq 0$.

Proof. Assume $R \leq s \leq t$. We have

$$\begin{aligned} \mathcal{F}(x, t) - \mathcal{F}(x, s) &= p \left[\frac{1}{p} (f(x, t)t - f(x, s)s) - (F(x, t) - F(x, s)) \right] \\ &= p \left[\int_R^t \frac{f(x, \tau)}{t^{p-1}} \tau^{p-1} \, d\tau - \int_R^s \frac{f(x, \tau)}{s^{p-1}} \tau^{p-1} \, d\tau - \int_s^t \frac{f(x, \tau)}{\tau^{p-1}} \tau^{p-1} \, d\tau + \frac{f(x, t)}{pt^{p-1}} R^p - \frac{f(x, s)}{ps^{p-1}} R^p \right] \\ &= p \left[\int_s^t \left(\frac{f(x, t)}{t^{p-1}} - \frac{f(x, \tau)}{\tau^{p-1}} \right) \tau^{p-1} \, d\tau + \int_R^s \left(\frac{f(x, t)}{t^{p-1}} - \frac{f(x, s)}{s^{p-1}} \right) \tau^{p-1} \, d\tau + \frac{R^p}{p} \left(\frac{f(x, t)}{t^{p-1}} - \frac{f(x, s)}{s^{p-1}} \right) \right] \geq 0. \end{aligned}$$

The case $t \leq s \leq -R$ is similar.

Finally, by (f_*) we see that $\mathcal{F} \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. Hence the number

$$C_1 = 1 + \sup_{(x,t) \in \Omega \times [-R,R]} \mathcal{F}(x, t) - \inf_{(x,t) \in \Omega \times [-R,R]} \mathcal{F}(x, t)$$

is finite. With this $C_1 > 0$ it is easy to see that (2.3) holds. \square

Remark 2.4. For the case $p = 2$ the conclusion of Lemma 2.3 has been stated in [12, Remark 1.1] without proof. For the case where f is differentiable, the result can also be proved as follows. We only consider the case $t \geq R$. By the monotonicity of $\frac{f(x,t)}{t^{p-1}}$ we deduce

$$\begin{aligned} \frac{\partial \mathcal{F}(x, t)}{\partial t} &= \frac{\partial}{\partial t} (f(x, t)t - pF(x, t)) \\ &= t \frac{\partial f(x, t)}{\partial t} - (p-1)f(x, t) = t^p \frac{\partial}{\partial t} \left(\frac{f(x, t)}{t^{p-1}} \right) \geq 0. \end{aligned}$$

Hence $\mathcal{F}(x, t)$ is increasing in $t \geq R$.

Lemma 2.5. *Assume that (f_*) , (f_1) , (f_2) and (f) are satisfied; then Φ satisfies the condition (C).*

Proof. Part of the proof (up to Eq. (2.7)) is identical to that of [19, Lemma 2.2] (see also [9, Lemma 2.2]). We include it here for the reader's convenience. Let $\{u_n\}$ be a Cerami sequence of Φ . Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, it suffices to show that $\{u_n\}$ is bounded.

If $\{u_n\}$ is unbounded, up to a subsequence we may assume that for some $c \in \mathbb{R}$,

$$\Phi(u_n) \rightarrow c, \quad \|u_n\| \rightarrow \infty, \quad \|u_n\| \|\Phi'(u_n)\| \rightarrow 0. \tag{2.4}$$

In particular

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left(\frac{1}{p} f(x, u_n) u_n - F(x, u_n) \right) \, dx = \lim_{n \rightarrow \infty} \left\{ \Phi(u_n) - \frac{1}{p} \langle \Phi'(u_n), u_n \rangle \right\} = c. \tag{2.5}$$

Let $w_n = \|u_n\|^{-1} u_n$; up to a subsequence we have

$$w_n \rightharpoonup w \text{ in } W_0^{1,p}(\Omega), \quad w_n \rightarrow w \text{ in } L^p(\Omega), \quad w_n(x) \rightarrow w(x) \text{ a.e. } x \in \Omega.$$

If $w = 0$, as in [11,15], we choose a sequence $\{t_n\} \subset [0, 1]$ such that

$$\Phi(t_n u_n) = \max_{t \in [0,1]} \Phi(t u_n).$$

For any $m > 0$, let $v_n = (2pm)^{1/p} w_n$. Since $v_n \rightarrow 0$ in $L^q(\Omega)$ and

$$|F(x, t)| \leq C(1 + |t|^q),$$

by the continuity of the Nemitskii operator, we see that $F(\cdot, v_n) \rightarrow 0$ in $L^1(\Omega)$. Thus

$$\lim_{n \rightarrow \infty} \int_{\Omega} F(x, v_n) \, dx = 0.$$

So for n large enough, $(2pm)^{1/p} \|u_n\|^{-1} \in (0, 1)$, and we deduce

$$\Phi(t_n u_n) \geq \Phi(v_n) = 2m - \int_{\Omega} F(x, v_n) \, dx \geq m. \quad (2.6)$$

That is, $\Phi(t_n u_n) \rightarrow \infty$. Now $\Phi(\mathbf{0}) = 0$, $\Phi(u_n) \rightarrow c$, we see that $t_n \in (0, 1)$, and

$$\int_{\Omega} |\nabla(t_n u_n)|^p \, dx - \int_{\Omega} f(x, t_n u_n) t_n u_n \, dx = \langle \Phi'(t_n u_n), t_n u_n \rangle = t_n \frac{d}{dt} \Big|_{t=t_n} \Phi(t u_n) = 0. \quad (2.7)$$

Therefore, using (2.3) we deduce

$$\begin{aligned} \int_{\Omega} \left(\frac{1}{p} f(x, u_n) u_n - F(x, u_n) \right) \, dx &\geq \int_{\Omega} \left(\frac{1}{p} f(x, t_n u_n) t_n u_n - F(x, t_n u_n) \right) \, dx - \frac{C_1}{p} |\Omega| \\ &= \frac{1}{p} \int_{\Omega} |\nabla(t_n u_n)|^p \, dx - \int_{\Omega} F(x, t_n u_n) \, dx - \frac{C_1}{p} |\Omega| \\ &= \Phi(t_n u_n) - \frac{C_1}{p} |\Omega| \rightarrow +\infty, \end{aligned}$$

This contradicts (2.5).

If $w \neq 0$, then the set $\Theta = \{x \in \Omega : w(x) \neq 0\}$ has positive Lebesgue measure. For $x \in \Theta$ we have $|u_n(x)| \rightarrow \infty$. Hence by (f₁) we deduce

$$\frac{F(x, u_n(x))}{|u_n(x)|^p} |w_n(x)|^p \rightarrow +\infty. \quad (2.8)$$

Since $\Phi(u_n) \rightarrow c$, using (2.8) and condition (f), we deduce via the Fatou Lemma that

$$\begin{aligned} \frac{1}{p} - \frac{c + o(1)}{\|u_n\|^p} &= \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|^p} \, dx \\ &= \left(\int_{w \neq 0} + \int_{w=0} \right) \frac{F(x, u_n)}{|u_n|^p} |w_n|^p \, dx \\ &\geq \int_{w \neq 0} \frac{F(x, u_n)}{|u_n|^p} |w_n|^p \, dx - \Lambda \int_{w=0} |w_n|^p \, dx \rightarrow +\infty. \end{aligned} \quad (2.9)$$

This is impossible.

In any case, we obtain a contradiction. Therefore, $\{u_n\}$ is bounded. \square

Remark 2.6. Note that under the assumptions of Theorems 1.1 and 1.2, the condition (f) is satisfied; hence Φ satisfies the condition (C).

For the proof of Theorem 1.2, we may assume that Φ has only finitely many critical points. Since Φ satisfies the condition (C), the critical groups $C_*(\Phi, \infty)$ of Φ at infinity make sense.

Lemma 2.7. Assume that (f_{*}), (f₁), (f₂) and (f) are satisfied; then $C_k(\Phi, \infty) \cong 0$ for all $k \in \mathbb{N}$.

Proof. To simplify the notation we define $X = W_0^{1,p}(\Omega)$. Let

$$S = \{u \in X : \|u\| = 1\}.$$

By (1.2) it is easy to see that for any $u \in S$, we have

$$\Phi(tu) \rightarrow -\infty, \quad \text{as } t \rightarrow +\infty.$$

Let $s = 0$ in (2.3); we see that

$$\mathcal{F}(x, t) \geq -C_1, \quad (x, t) \in \Omega \times \mathbb{R}. \tag{2.10}$$

Choose

$$a < \min \left\{ \inf_{\|u\| \leq 1} \Phi(u), -\frac{1}{p} C_1 |\Omega| \right\}.$$

Then for any $u \in S$, there exists $t > 1$ such that $\Phi(tu) \leq a$. If

$$\Phi(tu) = \frac{t^p}{p} - \int_{\Omega} F(x, tu) \, dx \leq a,$$

using (2.10) we obtain

$$\begin{aligned} \frac{d}{dt} \Phi(tu) &= t^{p-1} - \int_{\Omega} u f(x, tu) \, dx \\ &\leq \frac{1}{t} \left\{ pa + \int_{\Omega} pF(x, tu) \, dx - \int_{\Omega} t u f(x, tu) \, dx \right\} \\ &= \frac{1}{t} \left\{ pa - \int_{\Omega} \mathcal{F}(x, tu) \, dx \right\} \\ &\leq \frac{1}{t} \{pa + C_1 |\Omega|\} < 0. \end{aligned}$$

Therefore, by the Implicit Function Theorem, there exists a unique $T \in C(S, \mathbb{R})$ such that $\Phi(T(u)u) = a$.

Using the function T , we can follow the argument in [5, Page 4] to construct a strong deformation retract from $X \setminus \{0\}$ to Φ_a , and deduce

$$C_k(\Phi, \infty) = H_k(X, \Phi_a) \cong H_k(X, X \setminus \{0\}) = 0. \quad \square$$

Remark 2.8. A result similar to Lemma 2.7 has been obtained (for $p = 2$) by Wang [7], under the condition (f_*) and (AR). This result was then generalized to general $p > 1$ by Liu [5]. With the condition (Je), a similar result can be found in [9, Lemma 2.4].

To conclude this section we state the Fountain Theorem of Bartsch [25, Theorem 2.5]; see also [26, Theorem 3.6].

Let X be a reflexive and separable Banach space. It is well known that there exists $\{v_n\}_{n \in \mathbb{N}} \subset X, \{\varphi_n\}_{n \in \mathbb{N}} \subset X^*$ such that

1. $\langle \varphi_n, v_m \rangle = \delta_{n,m}$, where $\delta_{n,m} = 1$ for $n = m$ and $\delta_{n,m} = 0$ for $n \neq m$.
2. $\overline{\text{span}} \{v_n; n \in \mathbb{N}\} = X, \overline{\text{span}}^{w^*} \{\varphi_n; n \in \mathbb{N}\} = X^*$.

Let $X_j = \mathbb{R}v_j$, then $X = \overline{\bigoplus_{j \geq 1} X_j}$. Now we define

$$Y_k = \bigoplus_{j=1}^k X_j, \quad Z_k = \overline{\bigoplus_{j \geq k} X_j}. \tag{2.11}$$

Then we have the following Fountain Theorem.

Theorem 2.9 (Fountain Theorem). Assume that $\Phi \in C^1(X, \mathbb{R})$ satisfies the Cerami condition (C), $\Phi(-u) = \Phi(u)$. If for almost every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

1. $b_k = \inf_{u \in Z_k, \|u\|=r_k} \Phi(u) \rightarrow +\infty, \text{ as } k \rightarrow \infty,$
2. $a_k = \max_{u \in Y_k, \|u\|=\rho_k} \Phi(u) \leq 0,$

then Φ has a sequence of critical points $\{u_k\}$ such that $\Phi(u_k) \rightarrow +\infty$.

Remark 2.10. In [25,26], the Fountain Theorem is established under the Palais–Smale (PS) condition. Since the Deformation Theorem is still valid under the Cerami condition, we see that like many critical point theorems, the Fountain Theorem is true under the Cerami condition.

3. The proofs of the theorems

In this section we prove our theorems and give some remarks for further results.

Proof of Theorem 1.1. It is well known that under the assumptions the functional Φ has the mountain pass geometry. Since Φ satisfies condition (C), the Mountain Pass Theorem (with the Cerami condition) yields the desired result. \square

Remark 3.1. In fact, using the standard truncated technique and the Strong Maximum Principle we can obtain a positive solution u_+ and a negative one u_- . If $p = 2$ and f is of class C^1 and so Φ is of class C^2 , then the critical groups of Φ at u_{\pm} can be computed:

$$C_k(\Phi, u_{\pm}) \cong \delta_{k,1} \mathcal{G}.$$

With this result and our computation of $C_*(\Phi, \infty)$ in Lemma 2.7, using Morse theory we can obtain one more nontrivial solution as in [7,22].

Proof of Theorem 1.2. Since $f(x, 0) = 0$, the zero function $\mathbf{0}$ is a trivial critical point of Φ . By [5, Lemma 2.3], the condition (f_0) implies that $C_1(\Phi, \mathbf{0}) \neq 0$, while according to Lemma 2.7, $C_k(\Phi, \infty) = 0$ for all $k \in \mathbb{N}$. Now the desired result follows from Proposition 2.1. \square

Remark 3.2. Again, for the case $p = 2$ we can obtain a better result. Let

$$\lambda_1 < \lambda_2 \leq \dots \leq \lambda_m < \lambda_{m+1} \leq \dots$$

be the eigenvalues of $-\Delta$ on $H_0^1(\Omega)$, counted with multiplicity. Instead of (f_0^2) we assume

(f_0^3) there exists $\rho > 0$ such that $\lambda_m t^2 \leq 2F(x, t) \leq \lambda_{m+1} t^2$ for $|t| \leq \rho$.

Assume in addition that (f_*) , (f_1) and (f_2) hold. Then the problem (1.1) has a nontrivial solution.

In fact, according to [27, Lemma 4.2] the condition (f_0^3) implies that Φ has a local linking at $\mathbf{0}$ with respect to some decomposition $H_0^1(\Omega) = Y \oplus Z$, with $\dim Y = m$. Then by [28, Theorem 2.1] we have $C_m(\Phi, \mathbf{0}) \neq 0$.

Assuming the condition (AR), a similar result can be found in [29, Theorem 4].

Proof of Theorem 1.4. For the reflexive and separable Banach space $X = W_0^{1,p}(\Omega)$, define Y_k, Z_k as in (2.11). We know that Φ satisfies the Cerami condition (C), and $\Phi(-u) = \Phi(u)$. It remains to verify the conditions (i) and (ii) of Theorem 2.9. The verification of (i) is quite standard (see e.g. the proof of [26, Theorem 3.7] for the case $p = 2$), and therefore is omitted here.

Verification of (ii). Since our nonlinearity may not satisfy the growth condition (1.4), the standard argument as in the proof of [26, Theorem 3.7] does not work. Hence we use an indirect argument as follows.

Assume that (ii) of Theorem 2.9 does not hold for some given k . Then there exists a sequence $\{u_n\} \subset Y_k$ such that

$$\|u_n\| \rightarrow \infty, \quad \Phi(u_n) \geq 0. \quad (3.1)$$

Let $w_n = \|u_n\|^{-1} u_n$; then $\|w_n\| = 1$. Since $\dim Y_k < \infty$, there exists $w \in Y_k \setminus \{\mathbf{0}\}$ such that up to a subsequence,

$$\|w_n - w\| \rightarrow 0, \quad w_n(x) \rightarrow w(x) \quad \text{a.e. } x \in \Omega.$$

If $w(x) \neq 0$ then $|u_n(x)| \rightarrow \infty$, and hence by condition (f_1) , it holds that

$$\frac{F(x, u_n(x))}{|u_n(x)|^p} |w_n(x)|^p \rightarrow +\infty.$$

Using (f_1) and (f) we have, similar to (2.9),

$$\int_{\Omega} \frac{F(x, u_n)}{\|u_n\|^p} dx \rightarrow +\infty.$$

Hence

$$\Phi(u_n) = \|u_n\|^p \left(\frac{1}{p} - \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|^p} dx \right) \rightarrow -\infty,$$

a contradiction to (3.1). \square

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