On the regularity of operators near a regular operator ☆

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Abstract

Using the Riesz theorem, we give a new proof that the linear operators near a regular operator are regular.

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1. Introduction.

Let X be a Banach space and Y be a normed linear space. Recall that $A \in \mathscr{L}(X, Y)$ is a regular operator if $A : X \to Y$ is invertible and the inverse operator $A^{-1} : Y \to X$ is bounded. A classical result in operator theory is the following.

Theorem 1.1. *Let $A \in \mathscr{L}(X, Y)$ be a regular operator. Then there is some $\varepsilon > 0$ such that if $B \in \mathscr{L}(X, Y)$ and $\|B - A\| < \varepsilon$, then B is regular.*

Let $\mathscr{R}(X, Y)$ denotes the set of all regular operators from X to Y. Then this theorem means that $\mathscr{R}(X, Y)$ is open in $\mathscr{L}(X, Y)$.

Theorem 1.1 plays a major role in various area in mathematics. For example, it is used in the proof of the inverse function theorem in Banach spaces; see, e.g., [1, Theorem 4.1.1]. In the traditional proof of Theorem 1.1, one considers the case $X = Y$ and uses the fact that if $T \in \mathscr{L}(X, X)$ and $\|T\| < 1$ then $1_X - T$ is regular [2, Theorem 17.1.2]. Here 1_X is the identity operator in X. The general case is reduced to the above setting by considering the operator $A^{-1}B$.

The above proof relies on the convergence of series in $\mathscr{L}(X, X)$. In this note, we provide a different proof, which is more geometric in nature, and illustrates another application of the Riesz theorem.

2. Proof of Theorem 1.1.

Since A is regular, we can let $\varepsilon = \|A^{-1}\|^{-1}$. Then

$$\varepsilon \|x\| \le \|Ax\|, \qquad \text{for all } x \in X.$$

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If $\|B - A\| < \varepsilon$, then $\delta = \varepsilon - \|B - A\| > 0$. For all $x \in X$ we have

$$\begin{aligned} \|Bx\| &= \|Ax - (Ax - Bx)\| \\ &\geq \|Ax\| - \|Ax - Bx\| \geq \delta \|x\| . \end{aligned} \tag{2.1}$$

This inequality implies that B is an injection.

Moreover, the range $\operatorname{Im} B$ is closed in Y. In fact, let $y_n = Bx_n$ be a sequence in $\operatorname{Im} B$ such that $y_n \to y$. By (2.1) it follows that x_n is a Cauchy sequence in X. Since X is a Banach space, $x_n \to x$ for some $x \in X$. Now it is easy to show that $y = Bx \in \operatorname{Im} B$.

If we can prove that $\operatorname{Im} B = Y$, then B is invertible and (2.1) implies that $B^{-1} : Y \to X$ is bounded and the proof is completed.

Assume for a contradiction that $\operatorname{Im} B \neq Y$. Since $\operatorname{Im} B$ is closed, by the Riesz theorem [2, Lemma 5.2.7], for any $n \in \mathsf{N}$, there exists $y_n \in Y$ such that $\|y_n\| = 1$ and

$$1 - \frac{1}{n} < \inf_{y \in \operatorname{Im} B} \|y_n - y\| . \tag{2.2}$$

Since A is regular, in particular surjective, there exists a (unique) $x_n \in X$ such that $y_n = Ax_n$, so

$$\|x_n\| = \|A^{-1} y_n\| \leq \|A^{-1}\| \, \|y_n\| = \|A^{-1}\|.$$

Noting that $Bx_n \in \operatorname{Im} B$, we deduce from (2.2) that

$$\begin{aligned} 1 - \frac{1}{n} < \inf_{y \in \operatorname{Im} B} \|y_n - y\| &\leq \|y_n - Bx_n\| \\ &= \|Ax_n - Bx_n\| \\ &\leq \|A - B\| \, \|x_n\| \leq \|B - A\| \, \|A^{-1}\|. \end{aligned}$$

Hence $\|B - A\| \geq \|A^{-1}\|^{-1} = \varepsilon$. This contradicts $\|B - A\| < \varepsilon$, and the proof is concluded.

References

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[2] P. D. Lax, *Functional Analysis*, Wiley-Interscience, New York, 2002.