

Multiple solutions for elliptic resonant problems

Shibo Liu*

Department of Mathematics, Xiamen University, Xiamen 361005,
People's Republic of China

(MS received 17 April 2007; accepted 14 November 2007)

Two non-trivial solutions for semilinear elliptic resonant problems are obtained via the Lyapunov–Schmidt reduction and the three-critical-points theorem. The difficulty that the variational functional does not satisfy the Palais–Smale condition is overcome by taking advantage of the reduction and a careful analysis of the reduced functional.

1. Introduction

Let Ω be a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$. We consider the following semilinear elliptic boundary-value problem:

$$\left. \begin{aligned} -\Delta u &= p(x, u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \right\} \quad (1.1)$$

where $p : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that, for some $A > 0$,

$$|p(x, t)| \leq A|t| \quad (1.2)$$

holds for all $t \in \mathbb{R}$ and $x \in \Omega$. In particular, if the following limits exist:

$$\lim_{t \rightarrow 0} \frac{p(x, t)}{t} = p_0, \quad \lim_{|t| \rightarrow \infty} \frac{p(x, t)}{t} = p_\infty, \quad (1.3)$$

then problem (1.1) is called asymptotically linear. This kind of problem has been of great interest since the pioneering work of Amann and Zehnder [2].

Let $0 < \lambda_1 < \lambda_2 < \dots$ denote the distinct eigenvalues of $-\Delta$ on $H_0^1(\Omega)$. It is well known that, if there exists an eigenvalue between p_0 and p_∞ , then, in general, (1.1) has a non-trivial solution. If the nonlinearity $p(x, t)$ crosses the first eigenvalue λ_1 , one can even obtain multiple solutions for the problem (see, for example, [1, 7]).

For the case when the nonlinearity crosses a higher eigenvalue, to obtain multiple solutions one needs more conditions on the nonlinearity. In [12], using Morse theory [6], Li and Willem obtain two non-trivial solutions of (1.1), assuming that $p \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies (1.3) with $p_\infty = \lambda_\ell$, $\partial_t p(x, t) \geq \gamma > \lambda_{\ell-1}$ for all $(x, t) \in \Omega \times \mathbb{R}$, and

$$|p(x, t) - \lambda_\ell t| \leq c(1 + |t|^\alpha), \quad \lim_{|t| \rightarrow \infty} \frac{1}{|t|^{2\alpha}} (P(x, t) - \frac{1}{2} \lambda_\ell t^2) = -\infty \quad (1.4)$$

*Present address: Department of Mathematics, Shantou University, Shantou 515063, People's Republic of China (liusb@stu.edu.cn).

for some $\alpha \in [0, 1)$. Here

$$P(x, t) = \int_0^t p(x, s) \, ds.$$

Note that when $\alpha = 0$, this is exactly the well known Landesman–Lazer condition [10].

Let $X = H_0^1(\Omega)$ be the Sobolev space with inner product

$$\langle u, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v \, dx.$$

It is well known that the (weak) solutions of (1.1) are exactly the critical points of the C^1 -functional $f : X \rightarrow \mathbb{R}$,

$$f(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} P(x, u) \, dx. \quad (1.5)$$

In [12] condition (1.4) is used to ensure that f satisfies the Palais–Smale (PS) condition. This is crucial in order to apply the variational methods.

In this paper, we shall improve the above result of Li and Willem [12]. More precisely, we shall prove the following theorem.

THEOREM 1.1. *Suppose that $p : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function that satisfies (1.2) and there exists a real number $\gamma > \lambda_{\ell-1}$ such that*

$$\frac{p(x, t) - p(x, s)}{t - s} \geq \gamma \quad (1.6)$$

for $t, s \in \mathbb{R}$ and $x \in \Omega$. Assume that

$$\lim_{|t| \rightarrow \infty} (P(x, t) - \frac{1}{2} \lambda_{\ell} t^2) = -\infty \quad (1.7)$$

and there exist $\delta > 0$ and $k \geq \ell$ such that

$$\frac{1}{2} \lambda_k t^2 \leq P(x, t) \leq \frac{1}{2} \lambda_{k+1} t^2 \quad (1.8)$$

for $|t| \leq \delta$ and $x \in \Omega$. Then problem (1.1) has at least two non-trivial solutions.

REMARK 1.2. We can improve theorem 1.1 by replacing the real number γ in (1.6) by a suitable function $\gamma(x)$ (see remark 3.1 for details).

Note that in theorem 1.1 we assume only that $p(x, t)$ is a Carathéodory function, and we do not require that the limits in (1.3) exist. Moreover, our assumption (1.7) is much weaker than (1.4). We point out that our local condition (1.8) near $t = 0$ is also weaker than those required in [12].

Another feature of our result is that, under our assumptions the functional f does not satisfy the PS condition. However, we will see that the variational methods still apply. This is motivated by [15]. In that paper, Liu *et al.* assume that there exists a real number $\beta < \lambda_{\ell+1}$ such that

$$\frac{p(x, t) - p(x, s)}{t - s} \leq \beta. \quad (1.9)$$

Under the additional conditions (1.8) and (1.7) with reversed sign, two non-trivial solutions are obtained. The condition (1.9) enables one to apply the Lyapunov–Schmidt reduction procedure and then to consider the reduced functional φ . Since φ is defined on a finite-dimensional space, it is not difficult to show that φ is coercive, and hence satisfies the PS condition. Then the famous three-critical-points theorem [4, 13] yields the conclusion.

In our theorem 1.1, the situation is more delicate. Although condition (1.6) allows us to apply the reduction method, we are now led to a reduced functional φ , which is defined on an infinite-dimensional space. This makes it difficult to verify the coerciveness and the PS condition for φ .

The Lyapunov–Schmidt reduction has been used by many authors to obtain multiple solutions of nonlinear elliptic equations. In addition to [15] cited above, we mention [3, 9, 18]. In order to apply the critical point theory, the compactness (such as the PS condition) of the reduced functional is crucial. In [3] this issue is settled by showing that if the original functional satisfies the PS condition, then so does the reduced functional; this fact is also used in [9]. In [18] it is shown that the reduced functional satisfies the Cerami (C) condition provided that the original functional also satisfies it. In contrast to all these works, by taking advantage of the reduction, our approach does not require that the original functional satisfies the PS condition or the C condition.

The present paper is organized as follows. In § 2, for the reader's convenience, we review the Lyapunov–Schmidt reduction method. We also prove that, under appropriate conditions, if the gradient of the original functional is a compact perturbation of the identity, then the gradient of the reduced functional is also of this form. In § 3 we show that the reduced functional φ is really coercive and satisfies the PS condition. Then we apply the three-critical-points theorem to give the proof of theorem 1.1.

2. The Lyapunov–Schmidt reduction

Initially, let us recall the so-called Lyapunov–Schmidt reduction method. Let X be a Hilbert space, X^- and X^+ be closed subspaces of X such that $X = X^- \oplus X^+$. Let $f : X \rightarrow \mathbb{R}$ be a C^1 -functional. Assume that there exists a real number $\kappa > 0$ such that

$$-\langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2 \quad (2.1)$$

for all $v \in X^-$ and $w_1, w_2 \in X^+$. Then, according to [5], there exists a continuous function $\psi : X^- \rightarrow X^+$ such that

$$f(v + \psi(v)) = \max_{w \in X^+} f(v + w).$$

Moreover, the functional $\varphi : X^- \rightarrow \mathbb{R}$ given by $\varphi(v) = f(v + \psi(v))$ is of class C^1 . An element $v \in X^-$ is a critical point of φ if and only if $v + \psi(v)$ is a critical point of f . Let $P_{\pm} : X \rightarrow X^{\pm}$ be the orthogonal projection from X to X^{\pm} ; we also note that

$$\nabla \varphi(v) = P_- \nabla f(v + \psi(v)), \quad P_+ \nabla f(v + \psi(v)) = 0$$

for all $v \in X^-$.

It is well known that it is convenient to verify the PS condition for a functional, provided that the gradient of the functional is a compact perturbation of the identity map. Therefore, we need the gradient of the reduced functional in this form. This is case holds, provided that appropriate conditions are satisfied. To achieve this goal, we first show that the reduction map ψ is bounded.

LEMMA 2.1. *Let $f \in C^1(X, \mathbb{R})$ satisfies (2.1). If $\nabla f : X \rightarrow X$ is bounded, that is, it maps bounded sets to bounded sets, then the function $\psi : X^- \rightarrow X^+$ is also bounded.*

Proof. First we note that $P_+ \nabla f(v + \psi(v)) = 0$; in particular,

$$\langle \nabla f(v + \psi(v)), \psi(v) \rangle = 0.$$

As in [5, p. 13], setting $w_1 = \psi(v)$ and $w_2 = 0$ in (2.1), we obtain

$$\begin{aligned} \kappa \|\psi(v)\|^2 &\leq -\langle \nabla f(v + \psi(v)) - \nabla f(v), \psi(v) \rangle \\ &= \langle \nabla f(v), \psi(v) \rangle \\ &\leq \|\nabla f(v)\| \|\psi(v)\|. \end{aligned}$$

The desired result follows from the above inequality and the boundedness of ∇f . \square

COROLLARY 2.2. *If $\nabla f : X \rightarrow X$ is bounded and there is a compact nonlinear operator $K : X \rightarrow X$ such that $\nabla f = \mathbf{1}_X - K$, then there is a compact operator $Q : X^- \rightarrow X^-$ such that $\nabla \varphi = \mathbf{1}_{(X^-)} - Q$.*

Proof. For $v \in X^-$, we have

$$\begin{aligned} \nabla \varphi(v) &= P_- \nabla f(v + \psi(v)) \\ &= P_- \{(v + \psi(v)) - K(v + \psi(v))\} \\ &= v - P_- K(v + \psi(v)). \end{aligned}$$

Let $Q : X^- \rightarrow X^-$ be defined by

$$Q(v) = P_- K(v + \psi(v)).$$

If $\{v_n\}$ is a bounded sequence in X^- , then, by lemma 2.1, $\{\psi(v_n)\}$ and hence $\{v_n + \psi(v_n)\}$ is bounded in X . By the compactness of K and the continuity of P_- , we see that Q is compact. \square

3. Proof of theorem 1.1

Now let X be the Sobolev space $H_0^1(\Omega)$ and let f be the functional introduced in (1.5). To prove theorem 1.1, we have to show that f has at least two non-zero critical points.

Decompose the space X as $X = X^- \oplus X^+$ with

$$X^- = \overline{\bigoplus_{i=\ell}^{\infty} \ker(-\Delta - \lambda_i)}, \quad X^+ = \bigoplus_{i=1}^{\ell-1} \ker(-\Delta - \lambda_i).$$

By condition (1.6), for $v \in X^-$ and $w_1, w_2 \in X^+$, since

$$\int_{\Omega} (w_1 - w_2)^2 \, dx \geq \frac{1}{\lambda_{\ell-1}} \int_{\Omega} |\nabla(w_1 - w_2)|^2 \, dx,$$

we obtain

$$\begin{aligned} & - \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \\ & \geq - \int_{\Omega} |\nabla(w_1 - w_2)|^2 \, dx + \gamma \int_{\Omega} (w_1 - w_2)^2 \, dx \\ & \geq \left(\frac{\gamma}{\lambda_{\ell-1}} - 1 \right) \int_{\Omega} |\nabla(w_1 - w_2)|^2 \, dx = \left(\frac{\gamma}{\lambda_{\ell-1}} - 1 \right) \|w_1 - w_2\|^2. \end{aligned}$$

That is, f satisfies (2.1) with $\kappa = \lambda_{\ell-1}^{-1} \gamma - 1$.

REMARK 3.1. According to [8, proposition 2] (see also [18, lemma 2.2]), if $\gamma \in C(\bar{\Omega})$ is such that $\gamma(x) \geq \lambda_{\ell-1}$ on Ω and $\gamma(x) > \lambda_{\ell-1}$ strictly on a subset of Ω with positive measure, then there exists $\kappa > 0$ such that

$$- \int_{\Omega} |\nabla w|^2 \, dx + \int_{\Omega} \gamma(x) w^2 \, dx \geq \kappa \int_{\Omega} |\nabla w|^2 \, dx$$

for all $w \in X^+$. Using this result, we can slightly modify the above argument to show that, after replacing the real number γ in our condition (1.6) by such a function $\gamma(x)$, f still satisfies (2.1). Therefore, in this case, the result of theorem 1.1 is still valid.

Applying the Lyapunov–Schmidt reduction procedure, we obtain a continuous map $\psi : X^- \rightarrow X^+$ and a C^1 -functional $\varphi : X^- \rightarrow \mathbb{R}$ as described in the last section. It suffices to find two non-zero critical points of φ .

As the first step, we show that φ is coercive. The natural idea is to consider a sequence $\{v_n\}$ in X^- with $\|v_n\| \rightarrow \infty$. Then the sequence $\{\|v_n\|^{-1} v_n\}$ is bounded in X^- and possesses a weakly convergent subsequence. However, since $\dim X^- = \infty$, the weak limit may be the zero element in X^- . This makes it difficult to prove that $\varphi(v_n) \rightarrow +\infty$.

To get around this difficulty, we consider f_1 , the restriction of f on X^- . Obviously, $f_1 \in C^1(X^-, \mathbb{R})$ and for $v, \phi \in X^-$ we have

$$\langle \nabla f_1(v), \phi \rangle = \langle \nabla f(v), \phi \rangle = \int_{\Omega} \nabla v \cdot \nabla \phi \, dx - \int_{\Omega} p(x, v) \phi \, dx.$$

Note also that $\nabla f_1(v) = P_- \nabla f(v)$ for all $v \in X^-$.

Although f may not satisfy the PS condition, we will show that f_1 does satisfy the condition. The following observation is crucial.

LEMMA 3.2. Assume that $p(x, t)$ satisfies (1.2). Let $\{v_n\}$ be a sequence in X^- such that $\nabla f_1(v_n) \rightarrow 0$ and $\|v_n\| \rightarrow \infty$. Define $v_n^0 = \|v_n\|^{-1} v_n$. Then there is a subsequence of $\{v_n^0\}$ that converges weakly to some point $v \in X^-$. Moreover, $v \neq \mathbf{0}$.

Proof. Since $\{v_n^0\}$ is bounded, clearly it has a subsequence that converges weakly to some point $v \in X^-$. Now we show that $v \neq \mathbf{0}$.

Without loss of generality, we assume that $v_n^0 \rightharpoonup v$. The compactness of the embedding

$$X^- \hookrightarrow X \hookrightarrow L^2(\Omega)$$

implies that $v_n^0 \rightarrow v$ in $L^2(\Omega)$. Since $\nabla f_1(v_n) \rightarrow 0$, using (1.2) we have

$$\begin{aligned} \|v_n\| &\geq \langle \nabla f_1(v_n), v_n \rangle \\ &= \int_{\Omega} \nabla v_n \cdot \nabla v_n \, dx - \int_{\Omega} p(x, v_n) v_n \, dx \\ &\geq \int_{\Omega} |\nabla v_n|^2 \, dx - \Lambda \int_{\Omega} v_n^2 \, dx \\ &= \|v_n\|^2 - \Lambda |v_n|_2^2 \end{aligned}$$

for large n , where $|\cdot|_2$ denotes the L^2 -norm. Multiplying by $\|v_n\|^{-2}$ on both sides, we obtain

$$\|v_n\|^{-1} \geq 1 - \Lambda |v_n^0|_2^2.$$

Since $|v_n^0|_2 \rightarrow |v|_2$ and $\|v_n\|^{-1} \rightarrow 0$, from the above inequality we obtain $|v|_2^2 \geq \Lambda^{-1}$. Therefore, $v \neq \mathbf{0}$. □

REMARK 3.3. In a similar manner one can prove that if $\{u_n\}$ is a sequence in X such that $\nabla f(u_n) \rightarrow 0$ and $\|u_n\| \rightarrow \infty$, then the normalization sequence $\{u_n^0\}$ converges weakly to some point $u \in X \setminus \{\mathbf{0}\}$.

LEMMA 3.4. *The functional f_1 satisfies the PS condition.*

Proof. It is well known that there is a compact nonlinear operator $K : X \rightarrow X$ such that $\nabla f : X \rightarrow X$ is of the form $\nabla f = \mathbf{1}_X - K$. Since $\nabla f_1(v) = P_- \nabla f(v)$, we see that ∇f_1 is also of the form $\mathbf{1}_{(X^-)} - \text{compact}$. Therefore, according to [16, proposition 2.2] it suffices to show that any PS sequence for f_1 is bounded.

Let $\{v_n\}$ be a PS sequence of f_1 , that is, $v_n \in X^-$ and

$$\sup_n |f_1(v_n)| < \infty, \quad \nabla f_1(v_n) \rightarrow 0. \tag{3.1}$$

If $\{v_n\}$ is not bounded, then by going to a subsequence if necessary, we may assume that $\|v_n\| \rightarrow \infty$. Thus, let $v_n^0 = \|v_n\|^{-1} v_n$. By lemma 3.2 we conclude that, up to a subsequence, $v_n^0 \rightharpoonup v$ for some $v \in X^- \setminus \{\mathbf{0}\}$. Let

$$\Theta = \{x \in \Omega : v(x) \neq 0\}.$$

Then Θ is of positive Lebesgue measure. For $x \in \Theta$ we have $|v_n(x)| \rightarrow \infty$. Hence, by (1.7) and the Fatou lemma,

$$\int_{\Theta} (\frac{1}{2} \lambda_{\ell} v_n^2 - P(x, v_n)) \, dx \rightarrow +\infty \quad \text{as } n \rightarrow \infty. \tag{3.2}$$

On the other hand, (1.7) also implies that there exists $R > 0$ such that

$$\frac{1}{2} \lambda_{\ell} t^2 - P(x, t) \geq 0 \quad \text{if } |t| \geq R. \tag{3.3}$$

According to (1.2) we can find some $C > \lambda_{\ell}$ such that

$$|P(x, t)| \leq C(1 + |t|^2) \quad \text{for } (x, t) \in \Omega \times \mathbb{R}. \tag{3.4}$$

From (3.3) and (3.4) we see that

$$\frac{1}{2}\lambda_\ell t^2 - P(x, t) \geq \beta := (\frac{1}{2}\lambda_\ell - C)R^2 - C \tag{3.5}$$

for $(x, t) \in \Omega \times \mathbb{R}$. Therefore,

$$\int_{\Omega \setminus \Theta} (\frac{1}{2}\lambda_\ell v_n^2 - P(x, v_n)) \, dx \geq \beta \cdot \mathcal{L}(\Omega \setminus \Theta), \tag{3.6}$$

where $\mathcal{L}(\Omega \setminus \Theta)$ denotes the Lebesgue measure of $\Omega \setminus \Theta$.

Finally, by (3.2) and (3.6) we see that

$$\begin{aligned} f_1(v_n) &= \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 \, dx - \int_{\Omega} P(x, v_n) \, dx \\ &\geq \int_{\Omega} (\frac{1}{2}\lambda_\ell v_n^2 - P(x, v_n)) \, dx \\ &= \int_{\Theta} (\frac{1}{2}\lambda_\ell v_n^2 - P(x, v_n)) \, dx + \int_{\Omega \setminus \Theta} (\frac{1}{2}\lambda_\ell v_n^2 - P(x, v_n)) \, dx \\ &\geq \int_{\Theta} (\frac{1}{2}\lambda_\ell v_n^2 - P(x, v_n)) \, dx + \beta \cdot \mathcal{L}(\Omega \setminus \Theta) \rightarrow +\infty. \end{aligned}$$

This contradicts with (3.1). So $\{v_n\}$ must be bounded, and f_1 satisfies the PS condition. □

We are now in a position to show that the reduced functional φ is coercive and satisfies the PS condition.

LEMMA 3.5. *Under the assumptions of theorem 1.1, the reduced functional φ is coercive and bounded from below. Moreover, φ satisfies the PS condition.*

Proof. According to (3.5), for any $v \in X^-$ we have

$$\begin{aligned} f_1(v) &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} P(x, v) \, dx \\ &\geq \int_{\Omega} (\frac{1}{2}\lambda_\ell v^2 - P(x, v)) \, dx \\ &\geq \beta \cdot \mathcal{L}(\Omega). \end{aligned}$$

So $f_1 : X^- \rightarrow \mathbb{R}$ is bounded from below. Since f_1 satisfies the PS condition, by a well-known result of Li [11] (see also [17, corollary 2.7]) we see that f_1 is coercive.

By the definition of φ , we have

$$\varphi(v) = \max_{w \in X^+} f(v + w) \geq f(v) = f_1(v).$$

From this inequality, we conclude that φ is coercive and bounded from below.

As mentioned previously, there exists a compact operator $K : X \rightarrow X$ such that $\nabla f = \mathbf{1}_X - K$. According to corollary 2.2, there is also a compact operator $Q : X^- \rightarrow X^-$ such that $\nabla \varphi = \mathbf{1}_{(X^-)} - Q$. Since φ is coercive, any PS sequence of φ is bounded. Applying [16, proposition 2.2] again, we conclude that φ satisfies the PS condition. □

Now we can give the proof of theorem 1.1.

Proof of theorem 1.1. Let

$$V = \bigoplus_{i=1}^k \ker(-\Delta - \lambda_i), \quad Z = \overline{\bigoplus_{i=k+1}^{\infty} \ker(-\Delta - \lambda_i)}.$$

According to [14, lemma 4.2], condition (1.8) implies that the functional $f : X \rightarrow \mathbb{R}$ has a local linking at $\mathbf{0}$ with respect to the decomposition $X = V \oplus Z$; that is, there exists $\rho > 0$ such that

$$\begin{aligned} f(v) &\leq 0 && \text{for } v \in V, \|v\| \leq \rho, \\ f(v) &\geq 0 && \text{for } v \in Z, \|v\| \leq \rho. \end{aligned}$$

Decompose the space $X^- = Y \oplus Z$ with

$$Y = \bigoplus_{i=\ell}^k \ker(-\Delta - \lambda_i).$$

We will show that φ has a local linking at $\mathbf{0}$ with respect to this decomposition.

In fact, for $v \in Z, \|v\| \leq \rho$, we have

$$\varphi(v) \geq f(v) \geq 0. \tag{3.7}$$

On the other hand, by the continuity of ψ , the map $v \mapsto v + \psi(v)$ from Y to $Y \oplus X^+$ is continuous. Since $Y \oplus X^+$ is finite dimensional, we may use the L^∞ -norm $|\cdot|_\infty$ and deduce that there exists $r \in (0, \rho)$ such that if $v \in Y, \|v\| \leq r$, then

$$|v + \psi(v)|_\infty \leq \delta.$$

Hence, by (1.8) we obtain

$$\begin{aligned} \varphi(v) &= f(v + \psi(v)) \\ &= \frac{1}{2} \int_{\Omega} |\nabla(v + \psi(v))|^2 dx - \int_{\Omega} P(x, v + \psi(v)) dx \\ &\leq \frac{1}{2} \int_{\Omega} |\nabla(v + \psi(v))|^2 dx - \frac{1}{2} \lambda_k \int_{\Omega} |v + \psi(v)|^2 dx \leq 0. \end{aligned}$$

This inequality and (3.7) together imply that φ has a local linking at $\mathbf{0}$ with respect to the decomposition $X^- = Y \oplus Z$.

Since φ is bounded from below and satisfies the PS condition, by applying the three-critical-points theorem [4, 13] we obtain two non-zero critical points v_1, v_2 of φ . Now $u_1 = v_1 + \psi(v_1)$ and $u_2 = v_2 + \psi(v_2)$ are two non-zero solutions of problem (1.1). The proof of theorem 1.1 is complete. \square

Acknowledgments

The author is grateful to the referee for reading this paper carefully and for valuable suggestions and comments. This work was supported by the National Natural Science Foundation of China (Grant no. 10601041).

References

- 1 S. Ahmad. Multiple nontrivial solutions of resonant and nonresonant asymptotically linear problems. *Proc. Am. Math. Soc.* **96** (1986), 405–409.
- 2 H. Amann and E. Zehnder. Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations. *Annali Scuola Norm. Sup. Pisa* **7** (1980), 539–603.
- 3 D. Arcoya and D. G. Costa. Nontrivial solutions for strongly resonant problem. *Diff. Integ. Eqns* **8** (1995), 151–159.
- 4 H. Brezis and L. Nirenberg. Remarks on finding critical points. *Commun. Pure Appl. Math.* **44** (1991), 939–963.
- 5 A. Castro. Reduction methods via minimax. In *Differential equations*, Lecture Notes in Mathematics, vol. 957, pp. 1–20 (Springer, 1982).
- 6 K. C. Chang. *Infinite dimensional Morse theory and multiple solution problems* (Birkhäuser, 1993).
- 7 K. C. Chang, S. J. Li and J. Q. Liu. Remarks on multiple solutions for asymptotically linear elliptic boundary value problems. *Topolog. Meth. Nonlin. Analysis* **3** (1994), 179–187.
- 8 D. G. Costa and A. S. Oliveira. Existence of solution for a class of semilinear elliptic problems at double resonance. *Bol. Soc. Brasil Mat.* **19** (1988), 21–37.
- 9 N. Hirano and T. Nishimura. Multiplicity results for semilinear elliptic problems at resonance and with jumping nonlinearities. *J. Math. Analysis Applic.* **180** (1993), 566–586.
- 10 E. Landesman and A. Lazer. Nonlinear perturbations of linear eigenvalues problem at resonance. *J. Math. Mech.* **19** (1970), 609–623.
- 11 S. J. Li. Some existence theorems of critical points and applications. Report IC/86/90. ICTP, Trieste (1986).
- 12 S. J. Li and M. Willem. Multiple solutions for asymptotically linear boundary value problems in which the nonlinearity crosses at least one eigenvalue. *Nonlin. Diff. Eqns Applic.* **5** (1998), 479–490.
- 13 J. Q. Liu and S. J. Li. Existence theorems of multiple critical points and their applications. *Kexue Tongbao* **17** (1984), 1025–1027.
- 14 J. Q. Liu and J. B. Su. Remarks on multiple nontrivial solutions for quasilinear resonant problems. *J. Math. Analysis Applic.* **258** (2001), 209–222.
- 15 S. Q. Liu, C. L. Tang and X. P. Wu. Multiplicity of nontrivial solutions of semilinear elliptic equations. *J. Math. Analysis Applic.* **249** (2000), 289–299.
- 16 M. Struwe. *Variational methods*, 2nd edn (Springer, 1996).
- 17 M. Willem. *Minimax theorems* (Birkhäuser, 1996).
- 18 W. M. Zou and J. Q. Liu. Multiple solutions for resonant elliptic equations via local linking theory and Morse theory. *J. Diff. Eqns* **170** (2001), 68–95.

(Issued 5 December 2008)