

Multiple Solutions for Elliptic Resonant Problems [☆]

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Abstract

Two nontrivial solutions for semilinear elliptic resonant problems are obtained via the Lyapunov-Schmidt reduction and the three critical point theorem. The difficulty that the variational functional does not satisfy (PS) is overcome by taking advantage of the reduction and a careful analysis of the reduced functional.

1. Introduction

Let Ω be a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$. We consider the following semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = p(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $p : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that, for some $\Lambda > 0$ there holds

$$|p(x, t)| \leq \Lambda |t| \quad (1.2)$$

for all $t \in \mathbb{R}$ and $x \in \Omega$. In particular, if the following limits exist:

$$\lim_{t \rightarrow 0} \frac{p(x, t)}{t} = p_0, \quad \lim_{|t| \rightarrow \infty} \frac{p(x, t)}{t} = p_\infty, \quad (1.3)$$

then the problem (1.1) is called asymptotically linear. This kind of problems have captured great interest since the pioneer work of Amann-Zehnder [2].

Let $0 < \lambda_1 < \lambda_2 < \dots$ denote the distinct eigenvalues of $-\Delta$ on $H_0^1(\Omega)$. It is well known that, if there exists an eigenvalue between p_0 and p_∞ , then in general (1.1) has a nontrivial solution. If the nonlinearity $p(x, t)$ crosses the first eigenvalue λ_1 , one can even obtain multiple solutions for the problem, see for instance [1, 7].

For the case that the nonlinearity crosses a higher eigenvalue, to obtain multiple solutions one needs more conditions on the nonlinearity. In a recent paper [12], using Morse theory [6], Li-Willem obtain two nontrivial solutions of (1.1), assuming that $p \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies (1.3) with $p_\infty = \lambda_\ell$, $\partial_t p(x, t) \geq \gamma > \lambda_{\ell-1}$ for all $(x, t) \in \Omega \times \mathbb{R}$, and

$$|p(x, t) - \lambda_\ell t| \leq c(1 + |t|^\alpha), \quad \lim_{|t| \rightarrow \infty} \frac{1}{|t|^{2\alpha}} \left(P(x, t) - \frac{\lambda_\ell}{2} t^2 \right) = -\infty \quad (1.4)$$

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for some $\alpha \in [0, 1)$. Here $P(x, t) = \int_0^t p(x, s) ds$. Note that when $\alpha = 0$, this is exactly the well known Landesman–Lazer condition [10].

Let $X = H_0^1(\Omega)$ be the Sobolev space with inner product

$$\langle u, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v \, dx.$$

It is well known that the (weak) solutions of (1.1) are exactly the critical points of the C^1 -functional $f : X \rightarrow \mathbb{R}$,

$$f(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} P(x, u) \, dx. \quad (1.5)$$

In [12] the condition (1.4) is used to ensure that f satisfies the Palais-Smale (PS) condition. This is crucial for applying the variational methods.

In this paper, we shall improve the above result of Li-Willem [12]. More precisely, we shall prove the following theorem.

Theorem 1.1. *Suppose that $p : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function that satisfies (1.2) and there exists a real number $\gamma > \lambda_{\ell-1}$ such that*

$$\frac{p(x, t) - p(x, s)}{t - s} \geq \gamma \quad (1.6)$$

for $t, s \in \mathbb{R}$ and $x \in \Omega$. Assume that

$$\lim_{|t| \rightarrow \infty} \left(P(x, t) - \frac{\lambda_{\ell}}{2} t^2 \right) = -\infty \quad (1.7)$$

and there exist $\delta > 0$ and $k \geq \ell$ such that

$$\frac{1}{2} \lambda_k t^2 \leq P(x, t) \leq \frac{1}{2} \lambda_{k+1} t^2 \quad (1.8)$$

for $|t| \leq \delta$ and $x \in \Omega$. Then the problem (1.1) has at least two nontrivial solutions.

Remark 1.2. We can improve Theorem 1.1 by replacing the real number γ in (1.6) by a suitable function $\gamma(x)$, see Remark 3.1 in Page 4 for the details.

Note that in Theorem 1.1 we only assume that $p(x, t)$ is a Carathéodory function, and we don't require that the limits in (1.3) exist. Moreover, our assumption (1.7) is much weaker than (1.4). We point out that our local condition (1.8) near $t = 0$ is also weaker than those required in [12].

Another feature of our result is that, under the assumptions the functional f does not satisfy the (PS) condition. However, we will see that the variational methods still apply. This is motivated by Liu-Tang-Wu [15]. In that paper, the authors assume that there exists a real number $\beta < \lambda_{\ell+1}$ such that

$$\frac{p(x, t) - p(x, s)}{t - s} \leq \beta. \quad (1.9)$$

Under the additional conditions (1.8) and (1.7) with a reversed sign, two nontrivial solutions are obtained. The condition (1.9) enables one to apply the Lyapunov-Schmidt reduction procedure and turn to consider the reduced functional φ . Since φ is defined on a finite dimensional space, it is not difficult to show that φ is coercive, hence satisfies the (PS) condition. Then the famous three critical point theorem [4, 13] yields the conclusion.

In our Theorem 1.1, the situation is more delicate. Although the condition (1.6) allows us to apply the reduction method, we are now led to a reduced functional φ , which is defined on an infinite dimensional space. This makes it difficult to verify the coerciveness and (PS) for φ .

The Lyapunov-Schmidt reduction has been used by many authors to obtain multiple solutions of nonlinear elliptic equations. In addition to [15] above, we mention [3, 9, 18]. To apply the critical point theory the compactness (such as (PS) condition) of the reduced functional is crucial. In [3] this issue is settled by showing that if the original functional satisfies the (PS) condition, then so does the reduced functional; this fact is also used in [9]. In [18] it is shown that the reduced functional satisfies the Cerami condition (C) provided the original functional does. In contrast to all these works, by taking advantage of the reduction our approach does not require that the original functional satisfies the (PS) condition or the Cerami condition (C) .

The present paper is organized as follow. In section 2, to the reader's convenience we review the Lyapunov-Schmidt reduction method. We also prove that, under appropriate conditions if the gradient of the original functional is a compact perturbation of the identity, then the gradient of the reduced functional is also of this form. In section 3 we show that the reduced functional φ is really coercive and satisfies the (PS) condition. Then we apply the three critical point theorem to give the proof of Theorem 1.1.

2. The Lyapunov-Schmidt reduction

To start with, let us recall the so-called Lyapunov-Schmidt reduction method. Let X be a Hilbert space, X^- and X^+ be closed subspaces of X such that $X = X^- \oplus X^+$. Let $f : X \rightarrow \mathbb{R}$ be a C^1 -functional. Assume that there is a real number $\kappa > 0$ such that

$$-\langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2 \quad (2.1)$$

for all $v \in X^-$ and $w_1, w_2 \in X^+$. Then, according to [5], there exists a continuous function $\psi : X^- \rightarrow X^+$ such that

$$f(v + \psi(v)) = \max_{w \in X^+} f(v + w).$$

Moreover, the functional $\varphi : X^- \rightarrow \mathbb{R}$ given by $\varphi(v) = f(v + \psi(v))$ is of class C^1 . An element $v \in X^-$ is a critical point of φ if and only if $v + \psi(v)$ is a critical point of f . Let $P_{\pm} : X \rightarrow X^{\pm}$ be the orthogonal projection from X to X^{\pm} , we also note that

$$\nabla \varphi(v) = P_- \nabla f(v + \psi(v)), \quad P_+ \nabla f(v + \psi(v)) = 0$$

for all $v \in X^-$.

It is well known that it is convenient to verify the (PS) condition for a functional provided the gradient of the functional is a compact perturbation of the identity map. Therefore, we wish that the gradient of the reduced functional is of this form. This is really the case, provided appropriate conditions are satisfied. To achieve this goal, we first show that the reduction map ψ is bounded.

Lemma 2.1. *Let $f \in C^1(X, \mathbb{R})$ satisfies (2.1). If $\nabla f : X \rightarrow X$ is bounded, that is, it maps bounded sets to bounded sets, then the function $\psi : X^- \rightarrow X^+$ is also bounded.*

Proof. First we note that $P_+ \nabla f(v + \psi(v)) = 0$, in particular,

$$\langle \nabla f(v + \psi(v)), \psi(v) \rangle = 0.$$

As in [5, Page 13], setting $w_1 = \psi(v)$ and $w_2 = 0$ in (2.1) we obtain

$$\kappa \|\psi(v)\|^2 \leq -\langle \nabla f(v + \psi(v)) - \nabla f(v), \psi(v) \rangle$$

$$\begin{aligned}
&= \langle \nabla f(v), \psi(v) \rangle \\
&\leq \|\nabla f(v)\| \|\psi(v)\|.
\end{aligned}$$

The desired result follows from the above inequality and the boundedness of ∇f . \square

Corollary 2.2. *If $\nabla f : X \rightarrow X$ is bounded and there is a compact nonlinear operator $K : X \rightarrow X$ such that $\nabla f = \mathbf{1}_X - K$, then there is a compact operator $Q : X^- \rightarrow X^-$ such that $\nabla\varphi = \mathbf{1}_{(X^-)} - Q$.*

Proof. For $v \in X^-$, we have

$$\begin{aligned}
\nabla\varphi(v) &= P_- \nabla f(v + \psi(v)) \\
&= P_- \{(v + \psi(v)) - K(v + \psi(v))\} \\
&= v - P_- K(v + \psi(v)).
\end{aligned}$$

Let $Q : X^- \rightarrow X^-$ be defined by

$$Q(v) = P_- K(v + \psi(v)).$$

If $\{v_n\}$ is a bounded sequence in X^- , then by Lemma 2.1, $\{\psi(v_n)\}$ and hence $\{v_n + \psi(v_n)\}$ is bounded in X . By the compactness of K and the continuity of P_- , we see that Q is compact. \square

3. Proof of Theorem 1.1

Now let X be the Sobolev space $H_0^1(\Omega)$ and f be the functional introduced in (1.5). To prove Theorem 1.1, we have to show that f has at least two nonzero critical points.

Decompose the space X as $X = X^- \oplus X^+$ with

$$X^- = \overline{\bigoplus_{i=\ell}^{\infty} \ker(-\Delta - \lambda_i)}, \quad X^+ = \bigoplus_{i=1}^{\ell-1} \ker(-\Delta - \lambda_i).$$

By the condition (1.6), for $v \in X^-$ and $w_1, w_2 \in X^+$, since

$$\int_{\Omega} (w_1 - w_2)^2 dx \geq \frac{1}{\lambda_{\ell-1}} \int_{\Omega} |\nabla(w_1 - w_2)|^2 dx,$$

we obtain

$$\begin{aligned}
& - \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \\
& \geq - \int_{\Omega} |\nabla(w_1 - w_2)|^2 dx + \gamma \int_{\Omega} (w_1 - w_2)^2 dx \\
& \geq \left(\frac{\gamma}{\lambda_{\ell-1}} - 1 \right) \int_{\Omega} |\nabla(w_1 - w_2)|^2 dx = \left(\frac{\gamma}{\lambda_{\ell-1}} - 1 \right) \|w_1 - w_2\|^2.
\end{aligned}$$

That is, f satisfies (2.1) with $\kappa = \lambda_{\ell-1}^{-1} \gamma - 1$.

Remark 3.1. According to [8, Proposition 2] (see also [18, Lemma 2.2]), if $\gamma \in C(\bar{\Omega})$ is such that $\gamma(x) \geq \lambda_{\ell-1}$ on Ω and $\gamma(x) > \lambda_{\ell-1}$ strictly on a subset of Ω with positive measure, then there exists $\kappa > 0$ such that

$$- \int_{\Omega} |\nabla w|^2 dx + \int_{\Omega} \gamma(x) w^2 dx \geq \kappa \int_{\Omega} |\nabla w|^2 dx$$

for all $w \in X^+$. Using this result, we can slightly modify the above argument to show that, after replacing the real number γ in our condition (1.6) by such a function $\gamma(x)$, f still satisfies (2.1). Therefore, in this case the result of Theorem 1.1 is still valid.

Applying the Lyapunov-Schmidt reduction procedure, we obtain a continuous map $\psi : X^- \rightarrow X^+$ and a C^1 -functional $\varphi : X^- \rightarrow \mathbb{R}$ as described in the last section. It suffices to find two nonzero critical points of φ .

As the first step, we show that φ is coercive. The natural idea is to consider a sequence $\{v_n\}$ in X^- with $\|v_n\| \rightarrow \infty$. Then the sequence $\{\|v_n\|^{-1} v_n\}$ is bounded in X^- and possesses a weakly convergent subsequence. However, since $\dim X^- = \infty$, the weak limit may be the zero element in X^- . This makes it difficult to prove $\varphi(v_n) \rightarrow +\infty$.

To go around this difficulty, we consider f_1 , the restriction of f on X^- . Obviously, $f_1 \in C^1(X^-, \mathbb{R})$, for $v, \phi \in X^-$ we have

$$\langle \nabla f_1(v), \phi \rangle = \langle \nabla f(v), \phi \rangle = \int_{\Omega} \nabla v \cdot \nabla \phi \, dx - \int_{\Omega} p(x, v) \phi \, dx.$$

Note also that $\nabla f_1(v) = P_- \nabla f(v)$ for all $v \in X^-$.

Although f may not satisfy the (PS) condition, we will show that f_1 does satisfy (PS) . The following observation is crucial.

Lemma 3.2. *Assume that $p(x, t)$ satisfies (1.2). Let $\{v_n\}$ be a sequence in X^- such that $\nabla f_1(v_n) \rightarrow 0$ and $\|v_n\| \rightarrow \infty$. Denote $v_n^0 = \|v_n\|^{-1} v_n$, then there is a subsequence of $\{v_n^0\}$ that converges weakly to some point $v \in X^-$. Moreover, $v \neq \mathbf{0}$.*

Proof. Since $\{v_n^0\}$ is bounded, clearly it has a subsequence that converges weakly to some point $v \in X^-$. Now we show that $v \neq \mathbf{0}$.

Without loss of generality, we assume that $v_n^0 \rightharpoonup v$. The compactness of the embedding

$$X^- \hookrightarrow X \hookrightarrow L^2(\Omega)$$

implies that $v_n^0 \rightarrow v$ in $L^2(\Omega)$. Since $\nabla f_1(v_n) \rightarrow 0$, using (1.2) we have

$$\begin{aligned} \|v_n\| &\geq \langle \nabla f_1(v_n), v_n \rangle = \int_{\Omega} \nabla v_n \cdot \nabla v_n \, dx - \int_{\Omega} p(x, v_n) v_n \, dx \\ &\geq \int_{\Omega} |\nabla v_n|^2 \, dx - \Lambda \int_{\Omega} v_n^2 \, dx \\ &= \|v_n\|^2 - \Lambda |v_n|_2^2 \end{aligned}$$

for large n , where $|\cdot|_2$ denotes the L^2 -norm. Multiplying $\|v_n\|^{-2}$ in both sides, we obtain

$$\|v_n\|^{-1} \geq 1 - \Lambda |v_n^0|_2^2.$$

Since $|v_n^0|_2 \rightarrow |v|_2$ and $\|v_n\|^{-1} \rightarrow 0$, from the above inequality we obtain $|v|_2^2 \geq \Lambda^{-1}$. Therefore $v \neq \mathbf{0}$. \square

Remark 3.3. In a similar manner one can prove that if $\{u_n\}$ is a sequence in X such that $\nabla f(u_n) \rightarrow 0$ and $\|u_n\| \rightarrow \infty$, then the normalization sequence $\{u_n^0\}$ converges weakly to some point $u \in X \setminus \{\mathbf{0}\}$.

Lemma 3.4. *The functional f_1 satisfies the (PS) condition.*

Proof. It is well known that there is a compact nonlinear operator $K : X \rightarrow X$ such that $\nabla f : X \rightarrow X$ is of the form $\nabla f = \mathbf{1}_X - K$. Since $\nabla f_1(v) = P_- \nabla f(v)$, we see that ∇f_1 is

also of the form $\mathbf{1}_{(X^-)}$ – compact. So, according to [16, Proposition 2.2] it suffices to show that any (PS) sequence for f_1 is bounded.

Let $\{v_n\}$ be a (PS) sequence of f_1 , that is, $v_n \in X^-$ and

$$\sup_n |f_1(v_n)| < \infty, \quad \nabla f_1(v_n) \rightarrow 0. \quad (3.1)$$

If $\{v_n\}$ is not bounded, going to a subsequence if necessary, we may assume that $\|v_n\| \rightarrow \infty$. Thus, let $v_n^0 = \|v_n\|^{-1} v_n$, by Lemma 3.2 we conclude that up to a subsequence, $v_n^0 \rightharpoonup v$ for some $v \in X^- \setminus \{0\}$. Let

$$\Theta = \{x \in \Omega : v(x) \neq 0\},$$

then Θ is of positive Lebesgue measure. For $x \in \Theta$ we have $|v_n(x)| \rightarrow \infty$. Hence by (1.7) and the Fatou Lemma,

$$\int_{\Theta} \left(\frac{1}{2} \lambda_{\ell} v_n^2 - P(x, v_n) \right) dx \rightarrow +\infty, \quad \text{as } n \rightarrow \infty. \quad (3.2)$$

On the other hand, (1.7) also implies that there exists $R > 0$ such that

$$\frac{1}{2} \lambda_{\ell} t^2 - P(x, t) \geq 0, \quad \text{if } |t| \geq R. \quad (3.3)$$

According to (1.2) we can find some $C > \lambda_{\ell}$ such that

$$|P(x, t)| \leq C(1 + |t|^2), \quad \text{for } (x, t) \in \Omega \times \mathbb{R}. \quad (3.4)$$

From (3.3) and (3.4) we see that

$$\frac{1}{2} \lambda_{\ell} t^2 - P(x, t) \geq \beta := \left(\frac{1}{2} \lambda_{\ell} - C \right) R^2 - C \quad (3.5)$$

for $(x, t) \in \Omega \times \mathbb{R}$. Therefore,

$$\int_{\Omega \setminus \Theta} \left(\frac{1}{2} \lambda_{\ell} v_n^2 - P(x, v_n) \right) dx \geq \beta \cdot \mathcal{L}(\Omega \setminus \Theta), \quad (3.6)$$

where $\mathcal{L}(\Omega \setminus \Theta)$ denotes the Lebesgue measure of $\Omega \setminus \Theta$.

Finally, by (3.2) and (3.6) we see that

$$\begin{aligned} f_1(v_n) &= \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 dx - \int_{\Omega} P(x, v_n) dx \\ &\geq \int_{\Omega} \left(\frac{1}{2} \lambda_{\ell} v_n^2 - P(x, v_n) \right) dx \\ &= \int_{\Theta} \left(\frac{1}{2} \lambda_{\ell} v_n^2 - P(x, v_n) \right) dx + \int_{\Omega \setminus \Theta} \left(\frac{1}{2} \lambda_{\ell} v_n^2 - P(x, v_n) \right) dx \\ &\geq \int_{\Theta} \left(\frac{1}{2} \lambda_{\ell} v_n^2 - P(x, v_n) \right) dx + \beta \cdot \mathcal{L}(\Omega \setminus \Theta) \rightarrow +\infty. \end{aligned}$$

This contradicts with (3.1). So $\{v_n\}$ must be bounded, and f_1 satisfies the (PS) conditions. \square

We are now in the position to show that the reduced functional φ is coercive and satisfies the (PS) condition.

Lemma 3.5. *Under the assumptions of Theorem 1.1, the reduced functional φ is coercive and bounded from below. Moreover, φ satisfies the (PS) condition.*

Proof. According to (3.5), for any $v \in X^-$ we have

$$\begin{aligned} f_1(v) &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} P(x, v) \, dx \\ &\geq \int_{\Omega} \left(\frac{1}{2} \lambda_{\ell} v^2 - P(x, v) \right) \, dx \geq \beta \cdot \mathcal{L}(\Omega). \end{aligned}$$

So $f_1 : X^- \rightarrow \mathbf{R}$ is bounded from below. Since f_1 satisfies the (PS) condition, by a well known result of S.J. Li [11], see also [17, Corollary 2.7], we see that f_1 is coercive.

By the definition of φ , we have

$$\varphi(v) = \max_{w \in X^+} f(v + w) \geq f(v) = f_1(v).$$

From this inequality, we conclude that φ is coercive and bounded from below.

As mentioned before, there is a compact operator $K : X \rightarrow X$ such that $\nabla f = \mathbf{1}_X - K$. According to Corollary 2.2, there is also a compact operator $Q : X^- \rightarrow X^-$ such that $\nabla \varphi = \mathbf{1}_{(X^-)} - Q$. Since φ is coercive, any (PS) sequence of φ is bounded. Applying [16, Proposition 2.2] again, we conclude that φ satisfies the (PS) condition. \square

Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let

$$V = \bigoplus_{i=1}^k \ker(-\Delta - \lambda_i), \quad Z = \overline{\bigoplus_{i=k+1}^{\infty} \ker(-\Delta - \lambda_i)}.$$

According to [14, Lemma 4.2], the condition (1.8) implies that the functional $f : X \rightarrow \mathbf{R}$ has a local linking at $\mathbf{0}$ with respect to the decomposition $X = V \oplus Z$, that is, there exists $\rho > 0$ such that

$$\begin{aligned} f(v) &\leq 0, & \text{for } v \in V, \|v\| \leq \rho, \\ f(v) &\geq 0, & \text{for } v \in Z, \|v\| \leq \rho. \end{aligned}$$

Decompose the space $X^- = Y \oplus Z$ with

$$Y = \bigoplus_{i=\ell}^k \ker(-\Delta - \lambda_i),$$

we will show that φ has a local linking at $\mathbf{0}$ with respect to this decomposition.

In fact, for $v \in Z$, $\|v\| \leq \rho$ we have

$$\varphi(v) \geq f(v) \geq 0. \quad (3.7)$$

On the other hand, by the continuity of ψ , the map $v \mapsto v + \psi(v)$ from Y to $Y \oplus X^+$ is continuous. Since $Y \oplus X^+$ is finite dimensional, we may use the L^∞ -norm $|\cdot|_\infty$ and deduce that there exists $r \in (0, \rho)$ such that if $v \in Y$, $\|v\| \leq r$, then

$$|v + \psi(v)|_\infty \leq \delta.$$

Hence by (1.8) we obtain

$$\begin{aligned} \varphi(v) &= f(v + \psi(v)) \\ &= \frac{1}{2} \int_{\Omega} |\nabla(v + \psi(v))|^2 \, dx - \int_{\Omega} P(x, v + \psi(v)) \, dx \end{aligned}$$

$$\leq \frac{1}{2} \int_{\Omega} |\nabla(v + \psi(v))|^2 dx - \frac{\lambda_k}{2} \int_{\Omega} |v + \psi(v)|^2 dx \leq 0.$$

This inequality and (3.7) together imply that φ has a local linking at $\mathbf{0}$ with respect to the decomposition $X^- = Y \oplus Z$.

Since φ is bounded from below and satisfies (PS) , applying the three critical point theorem [4, 13] we obtain two nonzero critical points v_1, v_2 of φ . Now $u_1 = v_1 + \psi(v_1)$ and $u_2 = v_2 + \psi(v_2)$ are two nonzero solutions of the problem (1.1). The proof of Theorem 1.1 is complete. \square

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