On ground states of superlinear p-Laplacian equations in R^N ☆

Shibo Liu

Department of Mathematics, Shantou University
Shantou 515063, P.R. China

Abstract

We consider superlinear p-Laplacian equations in R^N with a potential which is periodic or has a bounded potential well. Without assuming the Ambrosetti-Rabinowitz type condition and the monotonicity of the function $t \mapsto \frac{f(x,t)}{|t|^{p-1}}$, we prove the existence of ground states. Even for the case $p = 2$, our results extend the recent results of Li, Wang and Zeng [Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), 829–837].

Key words: p-Laplacian, p-superlinear, Cerami sequences, Lions Lemma, ground states.
1991 MSC: 35J60, 58E05.

1. Introduction

In this paper, we consider the following p-Laplacian equation ($1 < p < N$):

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = f(x,u), \\ u \in W^{1,p}(\mathsf{R}^N). \end{cases} \tag{1.1}$$

We assume that the potential $V(x)$ and the nonlinearity $f(x,u)$ satisfies the following conditions:

(V_1) $V \in C(\mathsf{R}^N)$, and $0 < \alpha \le V(x) \le \beta < +\infty$.

(f_1) $f \in C(\mathsf{R}^N \times \mathsf{R})$ is 1 -periodic in $x_1, \dots, x_N$, and

$$\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p^*-1}} = 0, \qquad \lim_{|t|\to\infty} \frac{F(x,t)}{|t|^p} = +\infty \tag{1.2}$$

uniformly in $x \in \mathsf{R}^N$, where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$; $p^* = Np/(N-p)$ for $N \ge 3$ and an arbitrary real number greater than p for $N = 1, 2$.

(f_2) $f(x,t) = o(|t|^{p-2}t)$ as $|t| \to 0$, uniformly in $x \in \mathsf{R}^N$.

(f_3) there exists $\theta \ge 1$ such that $\theta\mathcal{F}(x,t) \ge \mathcal{F}(x,st)$ for $(x,t) \in \mathsf{R}^N \times \mathsf{R}$ and $s \in [0,1]$, where $\mathcal{F}(x,t) = f(x,t)t - pF(x,t)$.

☆ Supported by National Natural Science Foundation of China (10601041).
Email address: `liusb@stu.edu.cn` (Shibo Liu).

J. Math. Anal. Appl., in press.

Note that if the nonlinearity is subcritical and p-superlinear, that is for some $C > 0$ and $q \in (p, p^*)$,

$$|f(x,t)| \le C(1 + |t|^{q-1}), \qquad \lim_{|t|\to\infty} \frac{f(x,t)t}{|t|^p} = +\infty, \tag{1.3}$$

then (1.2) is satisfied. Our condition (1.2) is slightly weaker. The condition (1.2) was first introduced by Liu and Wang [23] and then was used in [19].

The study of elliptic problems of the form (1.1) in the semilinear case $p = 2$, has been motivated in part by seraching standing waves for the nonlinear Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \tilde{V}(x)\psi - g(x,\psi), \tag{1.4}$$

namely, solutions of the form $\psi(t,x) = \exp(-\mathrm{i}Et/\hbar)u(x)$. Where m and $\hbar$ are positive constants and g satisfies

$$g(x, s\xi) = f(x,s)\xi, \qquad (x,s) \in \mathsf{R}^N \times \mathsf{R}, \quad \text{and } \xi \in \mathsf{C} \text{ with } |\xi| = 1$$

for some $f \in C(\mathsf{R}^N \times \mathsf{R})$. The quasilinear case $p \in (1, N)$ also arise in a lot of applications, such as image processing, non-Newtonian fluids, pseudo-plastic fluids, nonlinear elasticity and reaction–diffusions, see [8] for more details.

By conditions (f_1) and (f_2), we deduce that there exists a constant $C > 0$ such that

$$|F(x,t)| \le C(|t|^p + |t|^{p^*}). \tag{1.5}$$

Consequently, the energy functional $\Phi : W^{1,p}(\mathsf{R}^N) \to \mathsf{R}$,

$$\Phi(u) = \frac{1}{p}\int_{\mathsf{R}^N} (|\nabla u|^p + V(x)|u|^p)\,\mathrm{d}x - \int_{\mathsf{R}^N} F(x,u)\mathrm{d}x \tag{1.6}$$

is of class C^1. The critical points of Φ are weak solutions of (1.1).

In many study on p-superlinear elliptic equations, the following superlinear condition of Ambrosetti and Rabinowitz [3] is assumed:

(AR) there is $\mu > p$ such that $0 < \mu F(x,t) \le tf(x,t)$ for $x \in \mathsf{R}^N$ and $t \ne 0$.

For the semilinear problem on R^N, we refer to the classical works [5, 6, 17, 25], which are concerning standing waves solutions of the nonlinear Schrödinger equation (1.4) mentioned above. On the other hand, the quasilinear case is usually considered on a bounded domain, the reader may find some results in this direction in [4, 7, 9, 21, 24]. In recent years more and more attention is paid to the quasilinear elliptic equations setting on R^N. To name a few we mention [1, 2, 11]. We would also like to mention a recent paper [12] where a quasilinear problem on R^N involving a more general differential operator, namely the so-call $p(x)$-Laplacian, is considered.

The role of (AR) is to ensure the boundness of the Palais-Smale (PS) sequences of the functional Φ. This is very crucial in applying the critical point theory. However, there are many functions which are p-superlinear at infinity, but do not satisfy the condition (AR) for any $\mu > p$. In fact, (AR) implies that $F(x,t) \ge C|t|^\mu$ for some $C > 0$. Thus, for example the p-superlinear function

$$f(x,t) = |t|^{p-2}t\log(1 + |t|)$$

does not satisfy (AR). However, it satisfies our condition (f_1)–(f_3).

The purpose of this paper is to study the problem (1.1) without assuming (AR). Our basic assumptions on the nonlinearity are (f_1)–(f_3). Note that for the semilinear case $p = 2$, the condition (f_3) is due to Jeanjean [14]. To overcome the difficulty that the (PS) sequences of

Φ may be unbounded, he established a variant of Mountain Pass Lemma which asserts that a sequence of perturbed functionals possesses bounded (PS) sequences. In [16], this condition is also used with a Cerami type argument in singularly perturbed elliptic problems in R^N with autonomous nonlinearity. For quasilinear elliptic problems setting on a bounded domain, the condition (f_3) is also used in [22] to obtain infinitely many solutions and in [13] to compute the critical groups of Φ at infinity and obtain nontrivial solutions via Morse theory. It is also important to note that if $\frac{f(x,t)}{|t|^{p-1}}$ is increasing in t, then (f_3) is satisfied, see [22, Proposition 2.3] for a proof.

By condition (f_2), we have $f(x,0) = 0$. Therefore the zero function $u = \mathbf{0}$ is a trivial solution of (1.1). We shall investigate the existence of nontrivial solutions for the problem (1.1). First, we assume that the potential V satisfies the following periodic condition.

(V_2) $V(x)$ is 1-periodic in $x_1, \dots, x_N$.

Then we have the following theorem.

Theorem 1.1. *Suppose the conditions* (V_1), (V_2) *and* (f_1)–(f_3) *hold, then the problem* (1.1) *has a* ground state, *i.e. a nontrivial solution* v *such that*

$$\Phi(v) = \inf\left\{\Phi(u) : u \neq \mathbf{0} \text{ and } \Phi'(u) = 0\right\}.$$

As in [19], we can also consider the potential well case. We assume that the potential V satisfies the following condition:

$$V(x) < V_\infty := \lim_{|x|\to\infty} V(x) < \infty, \qquad \text{for all } x \in \mathsf{R}^N, \tag{V_2^*}$$

and the nonlinearity $f(x,t) = f(t)$ does not depend on x. That is, the problem is of the form

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = f(u), \\ u \in W^{1,p}(\mathsf{R}^N). \end{cases} \tag{1.7}$$

Then we have an analogue of Theorem 1.1. For the sake of simplicity, we only consider the case $N \geq 3$.

Theorem 1.2. *Let* $N \geq 3$. *If the conditions* (V_1), (V_2^*) *and* (f_1)–(f_3) *hold with* f *independent of* x, *then the problem* (1.7) *has a ground state.*

For the case $p = 2$, assuming in addition that f is of class C^1 and $\frac{f(x,t)}{|t|}$ is *strictly* increasing in t, the existence of ground state has been proved by Li, Wang and Zeng [19]. Since the solution there is obtained as a minimizer of Φ on the Nehari manifold $\mathcal{N}$, it is crucial to require that f is of class C^1. Otherwise $\mathcal{N}$ may not be a smooth submanifold of the Sobolev space and it is not clear that the minimizer on $\mathcal{N}$ is a critical point of Φ. Our Theorems 1.1 and 1.2 do not require this smoothness condition. Therefore, even for the case $p = 2$ our results extend those of Li, Wang and Zeng [19].

Instead of minimizing the functional Φ on the Nehari manifold $\mathcal{N}$, we will prove our results by an mountain pass type argument. A crucial step is to prove that some kind of almost critical sequence is bounded. In Lemma 2.3 we adopt a technique developed in [14, 16] and the Lions Lemma [20, Lemma I.1] to show that any Cerami sequence of Φ is bounded. Then we obtain nontrivial solution by taking advantage of the Z^N-invariant of the problem for the periodic case and by energy comparison method for the potential well case. Finally, we show the existence of ground states by using a technique of Jeanjean and Tanaka [15, Theorem 4.5], where for the case

$p = 2$, in the potential well case, an asymptotically linear problem with autonomous nonlinearity is considered.

2. Proof of Theorem 1.1

Let X be a real Banach space, $\Phi \in C^1(X, \mathsf{R})$. Recall that a sequence $\{u_n\} \subset X$ is called a Palais-Smale sequence of Φ at the level c, a $(PS)_c$ sequence for short, if $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$. A sequence $\{u_n\} \subset X$ is called a Cerami sequence of Φ at the level c, a $(C)_c$ sequence for short, if

$$\Phi(u_n) \to c, \qquad (1 + \|u_n\|) \left\|\Phi'(u_n)\right\| \to 0.$$

For simplicity, from now on we denote $X = W^{1,p}(\mathsf{R}^N)$. Using condition (V$_1$), we can introduce a new norm $\|\cdot\|$ on X as follow

$$\|u\| = \left(\int_{\mathsf{R}^N} (|\nabla u|^p + V(x)|u|^p)\,\mathrm{d}x\right)^{1/p}, \qquad u \in X. \tag{2.1}$$

It is well known that the new norm $\|\cdot\|$ is equivalent to the standard norm $\|\cdot\|_{1,p}$ on X . With the norm $\|\cdot\|$, the functional Φ can be written as

$$\Phi(u) = \frac{1}{p}\|u\|^p - \int_{\mathsf{R}^N} F(x,u)\mathrm{d}x.$$

As an easy consequence of (f$_1$) and (f$_2$), we have the following lemma.

Lemma 2.1. *There exists $r > 0$ and $\phi \in X$ such that $\|\phi\| > r$ and*

$$b := \inf_{\|u\|=r} \Phi(u) > \Phi(\mathbf{0}) = 0 \geq \Phi(\phi). \tag{2.2}$$

Remark 2.2. In fact, (f$_2$) implies that as $\|u\| \to 0$ we have

$$\langle \Phi'(u), u\rangle = \|u\|^p + o\left(\|u\|^p\right), \qquad \Phi(u) = \frac{1}{p}\|u\|^p + o\left(\|u\|^p\right).$$

Therefore, it is also easy to see that

(i) there exists $\rho_0 > 0$ such that for any nontrivial critical point u of Φ, there holds $\|u\| \geq \rho_0$.
(ii) For any $c > 0$, there exists $\rho_c > 0$ such that if $\Phi(u_n) \to c$, then $\|u_n\| \geq \rho_c$.

By Lemma 2.1 we see that Φ has a mountain pass geometry: that is, setting

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = \mathbf{0} \text{ and } \Phi(\gamma(1)) < 0\},$$

we have $\Gamma \neq \emptyset$. By a special version of the Mountain Pass Lemma (see [10]), for the mountain pass level

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)), \tag{2.3}$$

there exists a $(C)_c$ sequence $\{u_n\}$ for Φ. Moreover, by (2.2) we see that $c > 0$.

Next, we show that this $(C)_c$ sequence is bounded. Before that, we deduce from (f$_3$) that

$$f(x,t)t - pF(x,t) \geq 0, \qquad \text{for all } (x,t) \in \mathsf{R}^N \times \mathsf{R}.$$

Let $t > 0$. For $x \in \mathsf{R}^N$, we have

$$\frac{\partial}{\partial t}\left(\frac{F(x,t)}{t^p}\right) = \frac{t^p f(x,t) - pt^{p-1}F(x,t)}{t^{2p}} \geq 0. \tag{2.4}$$

By (f_2),

$$\lim_{t\to 0^+} \frac{F(x,t)}{t^p} = 0. \tag{2.5}$$

From (2.4) and (2.5) we conclude that $F(x,t) \geq 0$ for all $x \in \mathbb{R}^N$ and $t \geq 0$. Arguing similarly for the case $t \leq 0$, eventualy we obtain

$$F(x,t) \geq 0, \qquad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.6}$$

Now we are ready to prove the following lemma, which is inspired by [16, §3], where the boundness of Cerami sequences for the modified functional related to a singular perturbed elliptic problem is considered. The potential $V(x)$ there is different from ours, the nonlinearity there is autonomous and satisfies the stronger condition (1.3) with $p = 2$.

Lemma 2.3. *Suppose that* (V_1), (f_1), (f_2) *and* (f_3) *hold,* $c \in \mathbb{R}$*, then any* $(C)_c$ *sequence of* Φ *is bounded.*

Proof. Let $\{u_n\}$ be a $(C)_c$ sequence of Φ. If $\{u_n\}$ is unbounded, up to a subsequence we may assume that

$$\Phi(u_n) \to c, \qquad \|u_n\| \to \infty, \qquad \left\|\Phi'(u_n)\right\| \|u_n\| \to 0.$$

In particular,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(\frac{1}{p} f(x,u_n)u_n - F(x,u_n)\right) \mathrm{d}x = \lim_{n\to\infty} \left\{\Phi(u_n) - \frac{1}{p}\langle \Phi'(u_n), u_n\rangle\right\} = c. \tag{2.7}$$

Let $v_n = \|u_n\|^{-1} u_n$, then $\{v_n\}$ is bounded in X. We claim that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |v_n|^p \,\mathrm{d}x = 0. \tag{2.8}$$

Otherwise, for some $\delta > 0$, up to a subsequence we have

$$\sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |v_n|^p \,\mathrm{d}x \geq \delta > 0.$$

We can choose $\{z_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_2(z_n)} |v_n|^p \,\mathrm{d}x \geq \frac{\delta}{2}.$$

It is easy to see that the number of points in $\mathbb{Z}^N \cap B_2(z_n)$ is less than 4^N. So there exists $y_n \in \mathbb{Z}^N \cap B_2(z_n)$, such that

$$\int_{B_2(y_n)} |v_n|^p \,\mathrm{d}x \geq \kappa := \frac{\delta}{2\times 4^N} > 0. \tag{2.9}$$

Let $\tilde{v}_n = v_n(\cdot + y_n)$. Using condition ($V_1$) and (2.1), we have

$$\begin{aligned}
\|\tilde{v}_n\|^p &= \int_{\mathbb{R}^N} (|\nabla \tilde{v}_n|^p + V(x)|\tilde{v}_n|^p) \,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} (|\nabla \tilde{v}_n|^p + \beta|\tilde{v}_n|^p) \,\mathrm{d}x = \int_{\mathbb{R}^N} (|\nabla v_n|^p + \beta|v_n|^p) \,\mathrm{d}x \\
&\leq \frac{\beta}{\alpha} \int_{\mathbb{R}^N} (|\nabla v_n|^p + \alpha|v_n|^p) \,\mathrm{d}x
\end{aligned}$$

$$\leq \frac{\beta}{\alpha} \int_{\mathbb{R}^N} (|\nabla v_n|^p + V(x)|v_n|^p)\, dx = \frac{\beta}{\alpha}.$$

That is, $\{\tilde{v}_n\}$ is also bounded in X. Passing to a subsequence we have

$$\tilde{v}_n \to \tilde{v} \text{ in } L^p_{\text{loc}}(\mathbb{R}^N), \quad \tilde{v}_n(x) \to \tilde{v}(x) \text{ a.e. } x \in \mathbb{R}^N.$$

Since

$$\int_{B_2(0)} |\tilde{v}_n|^p\, dx = \int_{B_2(y_n)} |v_n|^p\, dx \geq \kappa > 0, \tag{2.10}$$

we see that $\tilde{v} \neq \mathbf{0}$. Let $\tilde{u}_n = \|u_n\|\, \tilde{v}_n$. If $\tilde{v}(x) \neq 0$ we have $|\tilde{u}_n(x)| \to +\infty$, and using (1.2) we obtain

$$\frac{F(x, \tilde{u}_n(x))}{|\tilde{u}_n(x)|^p} |\tilde{v}_n(x)|^p \to +\infty. \tag{2.11}$$

Since $f(x, u)$ is 1-periodic with respect to x, we have

$$\int_{\mathbb{R}^N} F(x, u_n) dx = \int_{\mathbb{R}^N} F(x, \tilde{u}_n) dx. \tag{2.12}$$

Since the set $\Theta = \{x \in \mathbb{R}^N : \tilde{v}(x) \neq 0\}$ has positive Lebesgue measure, using (2.11) we have

$$\begin{aligned} \frac{1}{p} - \frac{c + o(1)}{\|u_n\|^p} &= \int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^p}\, dx \\ &= \int_{\mathbb{R}^N} \frac{F(x, \tilde{u}_n)}{\|u_n\|^p}\, dx \\ &\geq \int_{\tilde{v} \neq 0} \frac{F(x, \tilde{u}_n)}{|\tilde{u}_n|^p} |\tilde{v}_n|^p\, dx \to +\infty. \end{aligned} \tag{2.13}$$

This is impossible. Therefore we have proved (2.8). Now applying the Lions Lemma [20, Lemma I.1] we obtain

$$v_n \to \mathbf{0} \quad \text{in } L^q(\mathbb{R}^N), \qquad \forall q \in (p, p^*). \tag{2.14}$$

Next, we shall derive a controdiction as follow. Given a real number $R > 0$, by (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x, Rt)| \leq \varepsilon(|t|^p + |t|^{p^*}) + C_\varepsilon |t|^q. \tag{2.15}$$

Since $\|v_n\| = 1$, there exists a constant $C_1 > 0$ such that

$$|v_n|_p^p + |v_n|_{p^*}^{p^*} \leq C_1.$$

Therefore, by (2.14) and (2.15),

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |F(x, Rv_n)|\, dx \leq \limsup_{n\to\infty} \left\{ \varepsilon \left(|v_n|_p^p + |v_n|_{p^*}^{p^*} \right) + C_\varepsilon |v_n|_q^q \right\} \leq \varepsilon C_1.$$

Since $\varepsilon > 0$ is arbitrary, we deduce

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, Rv_n) dx = 0. \tag{2.16}$$

As in [14, 28], we choose a sequence $\{t_n\} \subset [0, 1]$ such that

$$\Phi(t_n u_n) = \max_{t \in [0,1]} \Phi(t u_n). \tag{2.17}$$

Given $m > 0$. Since for n large enough we have $\sqrt[p]{2pm}\,\|u_n\|^{-1} \in (0,1)$, using (2.16) with $R = \sqrt[p]{2pm}$, we obtain

$$\Phi(t_n u_n) \geq \Phi((2pm)^{1/p} v_n) = 2m - \int_{\mathsf{R}^N} F(x, (2pm)^{1/p} v_n)\mathrm{d}x \geq m.$$

That is, $\Phi(t_n u_n) \to +\infty$. But $\Phi(\mathbf{0}) = 0$, $\Phi(u_n) \to c$, using (2.17) we see that $t_n \in (0,1)$, and

$$\begin{aligned}\int_{\mathsf{R}^N} (|\nabla(t_n u_n)|^p + V(x)|t_n u_n|^p)\mathrm{d}x - \int_{\mathsf{R}^N} f(x, t_n u_n) t_n u_n \,\mathrm{d}x \\ = \langle \Phi'(t_n u_n), t_n u_n \rangle \\ = t_n \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_n} \Phi(t u_n) = 0.\end{aligned}$$

Now, by assumption (f_3),

$$\begin{aligned}&\int_{\mathsf{R}^N} \left(\frac{1}{p} f(x, u_n)u_n - F(x, u_n)\right)\mathrm{d}x \\ &\quad\geq \frac{1}{\theta}\int_{\mathsf{R}^N} \left(\frac{1}{p} f(x, t_n u_n) t_n u_n - F(x, t_n u_n)\right)\mathrm{d}x \\ &\quad= \frac{1}{\theta}\left(\frac{1}{p}\int_{\mathsf{R}^N} (|\nabla(t_n u_n)|^p + V(x)|t_n u_n|^p)\,\mathrm{d}x - \int_{\mathsf{R}^N} F(x, t_n u_n)\mathrm{d}x\right) \\ &\quad= \frac{1}{\theta}\Phi(t_n u_n) \to +\infty.\end{aligned}$$

This contradicts with (2.7). Therefore we have proved that $\{u_n\}$ is bounded. □

Remark 2.4. In the proof of Lemma 2.3, instead of (2.12), the existence of $\mu > 0$ such that

$$\int_{\mathsf{R}^N} F(x, u_n)\mathrm{d}x \geq \mu \int_{\mathsf{R}^N} F(x, \tilde{u}_n)\mathrm{d}x$$

is sufficient for deriving the contradiction (2.13). Therefore, it is not necessary to require $f(x,t)$ to be periodic in x. For example, the conclusion of Lemma 2.3 remains valid if we replace $f(x,t)$ with $b(x)f(t)$, where $b \in C(\mathsf{R}^N)$, $0 < b_0 \leq b(x) \leq b_1 < +\infty$ for all $x \in \mathsf{R}^N$.

Proof of Theorem 1.1. The proof consists of two steps.

Step 1. We use the standard argument (see e.g. [27, Page 106] for the case that (AR) holds and $p = 2$) to show that (1.1) has a nontrivial solution $\tilde{v}$. For $c > 0$ given in (2.3), there exists a $(C)_c$ sequence $\{u_n\}$ for the functional Φ. Using Lemma 2.3, the sequence $\{u_n\}$ is bounded in X. Let

$$\delta = \lim_{n\to\infty} \sup_{y \in \mathsf{R}^N} \int_{B_2(y)} |u_n|^p \,\mathrm{d}x. \tag{2.18}$$

If $\delta = 0$, applying the Lions Lemma [20, Lemma I.1] again,

$$u_n \to \mathbf{0} \quad \text{in } L^q(\mathsf{R}^N)\,, \qquad \forall q \in (p, p^*).$$

Thus, similar to (2.16), we have

$$\lim_{n\to\infty} \int_{\mathsf{R}^N} F(x, u_n)\mathrm{d}x = 0, \qquad \lim_{n\to\infty} \int_{\mathsf{R}^N} f(x, u_n)u_n \,\mathrm{d}x = 0. \tag{2.19}$$

Hence using (2.7) we have $c = 0$, a contradiction. Therefore $\delta > 0$. Similar to (2.10), we can find a sequence $\{y_n\} \subset \mathbb{Z}^N$ and a real number $\kappa > 0$ such that

$$\int_{B_2(0)} |v_n|^p \, \mathrm{d}x = \int_{B_2(y_n)} |u_n|^p \, \mathrm{d}x > \kappa, \tag{2.20}$$

where $v_n = u_n(\cdot + y_n)$. Note that $\|v_n\| = \|u_n\|$, we see that $\{v_n\}$ is bounded. Going if necessary to a subsequence, we obtain

$$v_n \rightharpoonup \tilde{v} \text{ in } X, \quad v_n \to \tilde{v} \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N);$$

and, by (2.20) we see that $\tilde{v} \neq \mathbf{0}$. Moreover, by the $\mathbb{Z}^N$ invariance of the problem, $\{v_n\}$ is also a $(C)_c$ sequence of Φ. Thus for every $\phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\langle \Phi'(\tilde{v}), \phi \rangle = \lim_{n\to\infty} \langle \Phi'(v_n), \phi \rangle = 0.$$

So $\Phi'(\tilde{v}) = 0$ and $\tilde{v}$ is a nontrivial solution of (1.1).

Step 2. Using the technique in the proof of [15, Theorem 4.5], where in the potential well case, an asymptotically linear problem with autonomous nonlinearity is considered, we show that the problem (1.1) has a ground state. Let

$$m = \inf \left\{ \Phi(u) : \, u \neq \mathbf{0} \text{ and } \Phi'(u) = 0 \right\}. \tag{2.21}$$

Assume that u is an arbitrary critical point of Φ. Since (f_3) implies

$$\mathcal{F}(x,t) \geq 0, \qquad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.22}$$

we deduce

$$\Phi(u) = \Phi(u) - \frac{1}{p} \langle \Phi'(u), u \rangle = \frac{1}{p} \int_{\mathbb{R}^N} \mathcal{F}(x,u) \mathrm{d}x \geq 0 \tag{2.23}$$

and thus $m \geq 0$. Therefore $0 \leq m \leq \Phi(\tilde{v}) < +\infty$. Let $\{u_n\}$ be a sequence of nontrivail critical points of Φ such that $\Phi(u_n) \to m$. According to Remark 2.2 (i) we see that

$$\|u_n\| \geq \rho_0 \tag{2.24}$$

for some $\rho_0 > 0$. Since u_n is critical we also have

$$(1 + \|u_n\|) \left\| \Phi'(u_n) \right\| \to 0.$$

Thus $\{u_n\}$ is a Cerami sequence at the level m. By Lemma 2.3, $\{u_n\}$ is bounded in X. For this sequence $\{u_n\}$, we denote δ as in (2.18). If $\delta = 0$, similar to (2.19) we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x,u_n) u_n \, \mathrm{d}x = 0,$$

and hence

$$\|u_n\|^p = \langle \Phi'(u_n), u_n \rangle + \int_{\mathbb{R}^N} f(x,u_n) u_n \, \mathrm{d}x \to 0. \tag{2.25}$$

This contradicts with (2.24). Hence $\delta > 0$. Now by the $\mathbb{Z}^N$ invariance of the problem, the same argument of Step 1 shows that a suitable translation of $\{u_n\}$, denoted by $\{v_n\}$, satisfies

$$\Phi'(v_n) = 0, \qquad \Phi(v_n) = \Phi(u_n) \to m,$$

and $\{v_n\}$ converges weakly to some $v \neq \mathbf{0}$, a nonzero critical point of Φ. Moreover, because of (2.22), using Fatou's lemma we deduce

$$\Phi(v) = \Phi(v) - \frac{1}{p} \langle \Phi'(v), v \rangle = \frac{1}{p} \int_{\mathbb{R}^N} \mathcal{F}(x,v) \mathrm{d}x$$

$$\le \liminf_{n\to\infty} \frac{1}{p} \int_{\mathbb{R}^N} \mathcal{F}(x, v_n)\mathrm{d}x$$
$$= \liminf_{n\to\infty} \left(\Phi(v_n) - \frac{1}{p}\langle \Phi'(v_n), v_n\rangle \right) = m. \tag{2.26}$$

Therefore, v is a nontrivial critical point of Φ with $\Phi(v) = m$, and Theorem 1.1 is proved. □

Remark 2.5. (i) Although v is a nontrivial solution, it is possible that $m = \Phi(v) = 0$. To obtain ground states with *positive* energy, in view of (2.23), it suffices to assume in addition that

$$\mathcal{F}(x,t) > 0, \qquad \text{for } x \in \mathbb{R}^N,\, t \neq 0.$$

This is the case if $\frac{f(x,t)}{|t|^{p-1}}$ is strictly increasing in t.

(ii) In general we only have $0 \le m \le c$. To guarantee $m = c$, we have to assume that $\frac{f(x,t)}{|t|^{p-1}}$ is strictly increasing in t. Then we can follow the standard argument (see e.g. [25, 27]) to show that

$$m = c = \inf\left\{\Phi(u) : u \neq \mathbf{0},\, \langle \Phi'(u), u\rangle = 0\right\} ,$$

except that instead of Palais-Smale sequences we should use the Cerami sequences which are bounded by Lemma 2.3. Here we don't need the condition (AR).

(iii) Observe that the truncated nonlinearity

$$f_+(x,u) = \begin{cases} f(x,u), & u \ge 0, \\ 0, & u < 0 \end{cases}$$

satisfies (f_1)–(f_3) except that the limits in (1.2) are now taken as $t \to \infty$. Thus, a parallel argument shows that the truncated problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = f_+(x,u), \\ u \in W^{1,p}(\mathbb{R}^N) \end{cases}$$

has a ground state v_+. By the strong maximum principle v_+ is a positive solution of (1.1) with least energy among all positive solutions.

3. Proof of Theorem 1.2

The proof of Theorem 1.2 goes as [15], where for the case $p = 2$, an asymptotically linear problem is considered. Recall that if $g \in C(\mathbb{R}^N)$,

$$|g(t)| \le C(|t|^{p-1} + |t|^{p^*-1}),$$

and $u \in W^{1,p}(\mathbb{R}^N)$ is a nontrivial solution of

$$-\,\mathrm{div}(|\nabla u|^{p-2}\nabla u) = g(u), \tag{3.1}$$

then we have the Pohozaev identity

$$\frac{N-p}{Np} \int_{\mathbb{R}^N} |\nabla u|^p\, \mathrm{d}x = \int_{\mathbb{R}^N} G(u)\mathrm{d}x, \tag{3.2}$$

where $G(t) = \int_0^t g(s)\mathrm{d}s$. The identity (3.2) follows from

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=1} \varphi(u^\lambda) = 0$$

with $u^\lambda(x) = u(\lambda x)$, $\lambda > 0$, and $\varphi \in C^1(X, \mathsf{R})$ given by

$$\varphi(u) = \frac{1}{p}\int_{\mathsf{R}^N} |\nabla u|^p \,\mathrm{d}x - \int_{\mathsf{R}^N} G(u)\mathrm{d}x.$$

For a nontrivial critical point $v \in X$ of φ, as in [15, §5] we set

$$v_t(x) = v(t^{-1}x) \tag{3.3}$$

for $t > 0$. By a direct calculation it is easy to see that v_t has the following properties:

$$\left.\begin{aligned} &|\nabla v_t|_p^p = t^{N-p}|\nabla v|_p^p, \\ &|v_t|_q^q = t^N|v|_q^q \text{ for all } q \in [p, p^*], \\ &\int_{\mathsf{R}^N} G(v_t)\mathrm{d}x = t^N \int_{\mathsf{R}^N} G(v)\mathrm{d}x. \end{aligned}\right\} \tag{3.4}$$

Using these properties, we can prove the following analogue of [15, Proposition 4.2].

Proposition 3.1. *If $v \in X$ is a nontrivial critical point of φ, then there exists $\gamma \in C([0,1], X)$ such that $\gamma(0) = \mathbf{0}$, $\varphi(\gamma(1)) < 0$, $v \in \gamma([0,1])$ and*

$$\max_{t\in[0,1]} \varphi(\gamma(t)) = \varphi(v).$$

Proof. The proof is similar to that of [15, Proposition 4.2]. Let v_t be given as in (3.3). By (3.4) and the Pohozaev identity (3.2) we have

$$\varphi(v_t) = \frac{t^{N-p}}{p}|\nabla v|_p^p - t^N \int_{\mathsf{R}^N} G(v)\mathrm{d}x = \left(\frac{1}{p}t^{N-p} - \frac{N-p}{Np}t^N\right)|\nabla v|_2^2.$$

We see that:

(i) $\max_{t>0} \varphi(v_t) = \varphi(v)$;
(ii) $\varphi(v_t) \to -\infty$ as $t \to \infty$;
(iii) $\|v_t\|_{1,p}^p = |\nabla v_t|_p^p + |v_t|_p^p = t^{N-2}|\nabla v|_p^p + t^N|v|_p^p \to 0$, as $t \to 0$.

Choose $\alpha > 1$ such that $\varphi(v_\alpha) < 0$ and set $\gamma(t) = v_{\alpha t}$ for $t \in (0,1]$ and $\gamma(0) = \mathbf{0}$, we obtain the desired γ. □

Proof of Theorem 1.2. We can now follow the approach presented in [15] to give the proof of Theorem 1.2. Since the nonlinearity does not depend on x, the functional Φ is now written as

$$\Phi(u) = \frac{1}{p}\int_{\mathsf{R}^N} (|\nabla u|^p + V(x)|u|^p)\mathrm{d}x - \int_{\mathsf{R}^N} F(u)\mathrm{d}x.$$

We also need to consider the functional $\Phi_\infty : X \to \mathsf{R}$,

$$\Phi_\infty(u) = \frac{1}{p}\int_{\mathsf{R}^N} (|\nabla u|^p + V_\infty|u|^p)\mathrm{d}x - \int_{\mathsf{R}^N} F(u)\mathrm{d}x.$$

Note that by condition (V_2^*), if $u \in X \setminus \{\mathbf{0}\}$, then

$$\Phi(u) < \Phi_\infty(u). \tag{3.5}$$

Step 1. We show that Φ has a nontrivial critical point. As before, for the mountain pass level $c > 0$, Φ has a $(C)_c$ sequence $\{u_n\}$. By Lemma 2.3, $\{u_n\}$ is bounded in X. Hence $u_n \rightharpoonup u$ in X, and u is a critical point of Φ. Assume that $u = \mathbf{0}$. Using condition (V_2^*) and the fact that $u_n \to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, we see that

$$|\Phi_\infty(u_n) - \Phi(u_n)| = \int_{\mathbb{R}^N} (V_\infty - V(x))|u_n|^p \, \mathrm{d}x \to 0,$$
$$\left\| \Phi'_\infty(u_n) - \Phi'(u_n) \right\| = \sup_{\phi \in X, \|\phi\|=1} \left| \int_{\mathbb{R}^N} (V_\infty - V(x)) u_n \phi \, \mathrm{d}x \right| \to 0.$$

That is, the bounded sequence $\{u_n\}$ is also a $(PS)_c$ sequence of Φ_∞. Let

$$\delta = \lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_2(y)} |u_n|^p \, \mathrm{d}x. \tag{3.6}$$

If $\delta = 0$, similar to (2.25) we have $\|u_n\|^p \to 0$. This is a contradiction with Remark 2.2 (ii). Therefore, $\delta > 0$ and there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_2(y_n)} |u_n|^p \, \mathrm{d}x \geq \frac{\delta}{2} > 0. \tag{3.7}$$

Let $v_n = u_n(\cdot + y_n)$. Since

$$\|v_n\| = \|u_n\|, \qquad \Phi_\infty(v_n) = \Phi_\infty(u_n), \qquad \Phi'_\infty(v_n) = \Phi'_\infty(u_n),$$

we see that $\{v_n\}$ is also a bounded $(PS)_c$ sequence of Φ_∞. As in Step 1 in the proof of Theorem 1.1, using (3.7) we see that $v_n \rightharpoonup v$ in X, $v \neq \mathbf{0}$ and v is a critical point of Φ_∞. Moreover, estimating as (2.26) we have

$$\Phi_\infty(v) \leq c.$$

Now applying Proposition 3.1 to Φ_∞, with

$$g(t) = f(t) - V_\infty t,$$

we see that there exists $\gamma \in C([0,1], X)$ such that $\gamma(0) = \mathbf{0}$, $\Phi_\infty(\gamma(1)) < 0$, $v \in \gamma([0,1])$ and

$$\max_{t\in[0,1]} \Phi_\infty(\gamma(t)) = \Phi_\infty(v).$$

By the construction of γ in the proof of Proposition 3.1, we also know that $\mathbf{0} \notin \gamma((0,1])$. Therefore according to (3.5) we have

$$\Phi(\gamma(t)) < \Phi_\infty(\gamma(t)) \tag{3.8}$$

for all $t \in (0,1]$. In particular $\Phi(\gamma(1)) \leq \Phi_\infty(\gamma(1)) < 0$ and hence $\gamma \in \Gamma$. Note that $\Phi(\mathbf{0}) = \Phi_\infty(\mathbf{0}) = 0$ and $c > 0$, we deduce

$$c \leq \max_{t\in[0,1]} \Phi(\gamma(t)) < \max_{t\in[0,1]} \Phi_\infty(\gamma(t)) = \Phi_\infty(v) \leq c. \tag{3.9}$$

This is a contradiction. Thus $u \neq \mathbf{0}$ and u is a nontrivial solution of problem (1.7). Note that since $\{u_n\}$ is a $(C)_c$ sequence of Φ and $u_n \rightharpoonup u$ in X, estimating as in (2.26) we have $\Phi(u) \leq c$.

Step 2. We show that the problem (1.7) has a ground state. Denote m as in (2.21). As before,

$$0 \leq m \leq \Phi(u) \leq c. \tag{3.10}$$

Now let $\{u_n\}$ be a sequence of nontrivial critical points of Φ such that $\Phi(u_n) \to m$. As in the proof of Theorem 1.1, $\{u_n\}$ is bounded and $\delta > 0$, where δ is defined via (3.6). Moreover, up to a subsequence $u_n \rightharpoonup \tilde{u}$ in X, and $\tilde{u}$ is a critical point of Φ, with $\Phi(\tilde{u}) \leq m$. If $\tilde{u} = \mathbf{0}$, as in Step 1 the bounded sequence $\{u_n\}$ is also a $(PS)_m$ sequence of Φ_∞. Since $\delta > 0$, a suitable translation of $\{u_n\}$, denoted by $\{v_n\}$, converges weakly to a nonzero critical point v of Φ_∞, with $\Phi_\infty(v) \le m$. Similar to (3.9), using Proposition 3.1, we can find a path $\gamma \in \Gamma$ such that

$$c \le \max_{t\in[0,1]} \Phi(\gamma(t)) < \max_{t\in[0,1]} \Phi_\infty(\gamma(t)) = \Phi_\infty(v) \le m.$$

This is a contradiction with (3.10). Therefore $\tilde{u} \neq \mathbf{0}$ is a nontrivial critical point of Φ with $\Phi(\tilde{u}) = m$. The proof is complete. □

Remark 3.2. In the proof of Theorem 1.2, the strict inequality (3.8) is crucial. For its validity, since we don't know whether the nontrivial critical point v of Φ_∞ can vanish on some set with positive Lebesgue measure, we have to assume $V(x) < V_\infty$ for all $x \in \mathsf{R}^N$.

If we are intended to find positive solutions of (1.7), then we are led to the truncated problem as described in Remark 2.5 (iii). In this case the nontrivial critical point v of the truncated Φ_∞ is positive everywhere, thus we only need to assume $V(x) \le V_\infty$ for all $x \in \mathsf{R}^N$, because this condition is sufficient for (3.8) in the case $V(x) \not\equiv V_\infty$, while the case $V(x) \equiv V_\infty$ is simpler.

In view of Remark 2.4, if we replace $f(t)$ with $b(x)f(t)$, where $b \in C(\mathsf{R}^N)$,

$$0 < b_0 := \lim_{|t|\to\infty} b(x) \le b(x) \le b_1 < +\infty \tag{3.11}$$

for all $x \in \mathsf{R}^N$, the Cerami sequences are still bounded. Working with the limiting functional

$$\Phi_\infty(u) = \frac{1}{p}\int_{\mathsf{R}^N} (|\nabla u|^p + V_\infty |u|^p)\mathrm{d}x - \int_{\mathsf{R}^N} b_0 F(u)\mathrm{d}x,$$

with little changes, we see that The conclusion of Theorem 1.2 remains valid.

4. Summary

The purpose of this paper is to investigate the existence of nontrivial solutions for superlinear p-Laplacian equations in R^N in the case that the nonlinearity may not satisfy the standard Ambrosetti-Rabinowitz condition (AR). Our results are new and improve some recent results in the literature even for the semilinear case $p = 2$, see the paragraph after Theorem 1.2.

The main ingredient of this work is Lemma 2.3. For the semilinear case $p = 2$ the boundedness of Cerami sequences has been proved in [15, §3], for the modified functional related to an autonomous problem. To deal with the nonautonomous case considered here, in the proof of Lemma 2.3 we have employed the argument between (2.9) and (2.13).

To conclude this paper, we state some further problems related to this work.

Problem 4.1. In (3.11), if $b_0 = 0$, then the limiting functional becomes

$$\Phi_\infty(u) = \frac{1}{p}\int_{\mathsf{R}^N} (|\nabla u|^p + V_\infty |u|^p)\mathrm{d}x.$$

This functional does not have nonzero critical points. Therefore the energy comparision method does not work. Thus it seems interesting to investigate the existence of solutions for this case.

Problem 4.2. For the semilinear case $p = 2$. It is well known that if $V \in C(\mathsf{R}^N)$ is periodic, then the Schrödinger operator $S : H^2(\mathsf{R}^N) \to L^2(\mathsf{R}^N)$,

$$u \mapsto -\Delta u + V(x)u$$

has purely continuous spectrum which is bounded below and consists of closed disjoint intervals, see e.g. [26, Thm XIII.100]. Under the assumption of Theorem 1.1, except that V is now allowed to take negative values. If 0 lies in a spectral gap of S, then the corresponding function Φ is strongly indefinite and the mountain pass lemma is not applicable. Naturally one may try to apply the *generalized linking theorem* [18, Theorem 2.1] to produce a Cerami sequence $\{u_n\}$ for Φ. Unfortunately the proof of Lemma 2.3 is not sufficient to obtain the boundeness of $\{u_n\}$: in the later part we can not deduce $\Phi(t_n u_n) \to +\infty$ because Φ is strongly indefinite. Therefore, it is interesting and challenging to prove the existence of nontrivial solutions for this case.

References

[1] C.O. Alves, G.M. Figueiredo, Existence and multiplicity of positive solutions to a p-Laplacian equation in $\mathbb{R}^N$, Differential Integral Equations 19 (2006), 143–162.

[2] C.O. Alves, G.M. Figueiredo, On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in $\mathbb{R}^N$, J. Differential Equations 246 (2009) 1288–1311.

[3] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973), 349–381.

[4] T. Bartsch, Z.L. Liu, On a superlinear elliptic p-Laplacian equation, J. Differential Equations 198 (2004), 149–175.

[5] T. Bartsch, Z.-Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, Comm. Partial Differential Equations 20 (1995), 1725–1741.

[6] V. Coti Zelati, P.H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^N$, Comm. Pure Appl. Math. XIV (1992), 1217–1269.

[7] M. Degiovanni, S. Lancelotti, Linking over cones and nontrivial solutions for p-Laplace equations with p-superlinear nonlinearity, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007), 907–919.

[8] J.I. Díaz, Nonlinear Partial Differential Equations and Free Boundaries, Vol. I. Elliptic Equations, Res. Notes Math., vol. 106, Pitman, Boston, MA, 1985, 323 pp.

[9] G. Dinca, P. Jebelean, J. Mawhin, Variational and topological methods for Dirichlet problems with p-Laplacian, Port. Math. (N.S.) 58 (2001), 339–378.

[10] I. Ekeland, Convexity Methods in Hamiltonian Mechanics, Springer, 1990.

[11] A. El Khalil, S. El Manouni and M. Ouanan, On some nonlinear elliptic problems for p-Laplacian in $\mathbb{R}^N$, NoDEA 15 (2008), 295–307.

[12] X.L. Fan, $p(x)$-Laplacian equations in $\mathbb{R}^N$ with periodic data and nonperiodic perturbations, J. Math. Anal. Appl. 341 (2008), 103–119.

[13] F. Fang, S.B. Liu, Nontrivial solutions of superlinear p-Laplacian equations, J. Math. Anal. Appl. 351 (2009), 138–146,

[14] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer type problem set on $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh 129 (1999), 787–809.

[15] L. Jeanjean, K. Tanaka, A positive solution for aysmptotically linear elliptic problem on $\mathbb{R}^N$ autonomous at infinity, ESAIM Control Optim. Calc. Var. 7 (2002), 597–614.

[16] L. Jeanjean, K. Tanaka, Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities, Calc. Var. 21 (2004), 287–318.

[17] W. Kryszewski, A. Szulkin, Generalized linking theorem with an application to a semilinear Schrödinger equation, Adv. Differential Equations 3 (1998), 441–472.

[18] G.B. Li, A. Szulkin, An asymptotically periodic Schrödinger equation with indefinite linear part, Commun. Contemp. Math. 4 (2002), 763–776.

[19] Y.Q. Li, Z.-Q. Wang, J. Zeng, Ground states of nonlinear Schr ödinger equations with potentials, Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), 829–837.

[20] P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 2, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), 223–283.

[21] S.B. Liu, Existence of solutions to a superlinear p-Laplacian equation. Electron, J. Differential Equations 2001, No. 66, 6 pp.

[22] S.B. Liu, S.J. Li, Infinitely many solutions for a superlinear elliptic equation, Acta Math. Sinica (Chin. Ser.) 46 (4) (2003), 625–630 (in Chinese).

[23] Z.L. Liu, Z.-Q. Wang, On the Ambrosetti-Rabinowitz superlinear condition, Adv. Nonlinear Stud. 4 (2004), 561–572.

[24] K. Perera, Nontrivial critical groups in p-Laplacian problems via the Yang index, Topol. Methods Nonlinear Anal. 21 (2003), 301–309.
[25] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992), 270–291.
[26] M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. IV, Academic Press, New York, 1978.
[27] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.
[28] W.M. Zou, Variant fountain theorems and their applications, Manuscripta Math. 104 (2001), 343–358.