

Nontrivial solutions of superlinear p -Laplacian equations \star

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Abstract

We consider p -Laplacian equations on a bounded domain, where the nonlinearity is superlinear but does not satisfy the usual Ambrosetti-Rabinowitz condition near infinity, or its dual version near zero. Nontrivial solutions are obtained by computing the critical groups and Morse theory.

Key words: p -Laplacian, Eigenvalues, Cerami condition, Critical groups, Fatou lemma
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1. Introduction

Let Ω be a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$. Consider the Dirichlet problem for the p -Laplacian equation ($p > 1$),

$$\begin{cases} -\Delta_p u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Here $-\Delta_p u$ is the p -Laplacian operator: $-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$.

Let $F(x, t) = \int_0^t f(x, s) ds$, $\mathcal{F}(x, t) = f(x, t)t - pF(x, t)$. We impose the following conditions on the nonlinearity $f(x, u)$.

(f_\star) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function and there exists some $q \in (p, p^*)$ such that for $(x, t) \in \Omega \times \mathbb{R}$ we have

$$|f(x, t)| \leq C(1 + |t|^{q-1});$$

where $p^* = Np/(N - p)$ if $N > p$, and $p^* = \infty$ if $N \leq p$.

(f_1) $f(x, t)t \geq 0$, and $f(x, t)$ is superlinear at infinity, that is, the following limit holds uniformly on $x \in \Omega$.

$$\lim_{|t| \rightarrow \infty} \frac{f(x, t)}{|t|^{p-2} t} = +\infty. \quad (1.2)$$

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(f_2) there exists $\theta \geq 1$ such that $\theta \mathcal{F}(x, t) \geq \mathcal{F}(x, st)$ for $(x, t) \in \Omega \times \mathbb{R}$ and $s \in [0, 1]$.
The condition (f_\star) implies that the functional $\Phi : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$,

$$\Phi(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} F(x, u) \, dx,$$

is well defined and of class C^1 . It is well known that the critical points of Φ are weak solutions of (1.1). In the last decades, many existence and multiplicity results were obtained by applying the critical point theory to Φ .

The condition (1.2) characterizes the problem (1.1) as superlinear at infinity. Such superlinear elliptic boundary value problem has been studied by many authors, see for instance [1,5,10,15,16]. A common feature of these works is that the following condition, which is originally due to Ambrosetti and Rabinowitz for the case $p = 2$ in [1], is imposed on the nonlinearity $f(x, t)$

(AR) There exists $\mu > p$ and $R > 0$ such that

$$0 < \mu F(x, t) \leq f(x, t)t, \quad \text{for } x \in \Omega \text{ and } |t| \geq R.$$

The role of (AR) is to ensure the boundness of the Palais-Smale sequences of the functional Φ . This is very crucial in the applications of critical point theory.

However, there are many functions which are superlinear at infinity, but does not satisfy the condition (AR) for any $\mu > p$. For example, the function

$$f(x, t) = |t|^{p-2} t \log(1 + |t|)$$

does not satisfy (AR), but it satisfies our conditions (f_\star), (f_1) and (f_2), see Remark 1.2 (iii).

Assume that $f(x, 0) = 0$, then the zero function $u = \mathbf{0}$ is a trivial solution of the problem (1.1). In this paper we shall investigate the existence of nontrivial solutions for the problem (1.1). For this purpose, some conditions on the nonlinearity near zero are in order.

Let λ_1 and λ_2 be the first and the second eigenvalues of $-\Delta_p$ on $W_0^{1,p}(\Omega)$. It is well known that $\lambda_1 > 0$ is a simple eigenvalue, and that $\sigma(-\Delta_p) \cap (\lambda_1, \lambda_2) = \emptyset$, where $\sigma(-\Delta_p)$ is the spectrum of $-\Delta_p$, (cf. [2]). Assume

(f_0) There exist $\rho > 0$, $\bar{\lambda} \in (\lambda_1, \lambda_2)$ such that

$$\lambda_1 |t|^p \leq pF(x, t) \leq \bar{\lambda} |t|^p, \quad \text{for } x \in \Omega \text{ and } |t| \leq \rho.$$

We are now ready to present our first result.

Theorem 1.1. *Assume that (f_\star), (f_0), (f_1) and (f_2) are satisfied, then the problem (1.1) has a nontrivial weak solution in $W_0^{1,p}(\Omega)$.*

Remark 1.2. (i) For the case $p \neq 2$, in the literature, to find nontrivial solutions most attention is paid to the case that $\mathbf{0}$ is a local minimizer of Φ , then the mountain pass theorem is applied. However, under our assumption (f_0), $\mathbf{0}$ is not a local minimizer of Φ . This case was considered in Liu [10], where a nontrivial solution is obtained under the condition (f_\star), (f_0) and (AR).

(ii) In the case $p = 2$, for a semilinear problem setting on \mathbb{R}^N , the condition (f_2) is originally due to Jeanjean [8]. Later, it was used by Liu-Li [11] for the general case $p > 1$. In [11], infinitely many solutions of (1.1) are obtained, provided that $f(x, u) = -f(x, -u)$ and (f_\star), (f_1) and (f_2) are satisfied.

(iii) It turns out that if for fixed $x \in \Omega$, $\frac{f(x, t)}{|t|^{p-2} t}$ is increasing in $(0, +\infty)$ and decreasing in $(-\infty, 0)$, then (f_2) is satisfied; see [11, Proposition 2.3] for a proof.

In (f_0) only the first two eigenvalues are involved. As in Perera [15], we may consider some situations involving higher eigenvalues. Assuming

(f_0^*) the limit $\lambda = \lim_{|t| \rightarrow 0} \frac{f(x, t)}{|t|^{p-2} t}$ exists uniformly in $x \in \Omega$,

we shall prove the following theorem.

Theorem 1.3. *Assume that (f_*) , (f_0^*) , (f_1) and (f_2) are satisfied. If $\lambda \notin \sigma(-\Delta_p)$, then the problem (1.1) has a nontrivial weak solution in $W_0^{1,p}(\Omega)$.*

Remark 1.4. In [15], Perera obtained a nontrivial solution under the conditions (f_*) , (f_0^*) and (AR).

In Theorems 1.1 and 1.3, the nonlinearity is superlinear at infinity and asymptotically linear at zero. It is natural to consider the dual case. In our last result we consider the case that the nonlinearity is asymptotically linear at infinity and superlinear at zero. We assume the following conditions on $f(x, u)$.

(f_3) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, and

$$\lim_{|t| \rightarrow \infty} \frac{f(x, t)}{|t|^{p-2} t} = \lambda$$

uniformly in $x \in \Omega$, for some $\lambda \in \mathbb{R}$.

(f_4) the following limit holds uniformly on $x \in \Omega$.

$$\lim_{|t| \rightarrow 0} \frac{F(x, t)}{|t|^p} = +\infty. \quad (1.3)$$

(f_5) $pF(x, t) - f(x, t)t > 0$ for all $x \in \Omega$ and $t \neq 0$.

We shall prove the following theorem.

Theorem 1.5. *Assume that (f_3) , (f_4) and (f_5) are satisfied. If $\lambda \notin \sigma(-\Delta_p)$, then the problem (1.1) has a nontrivial weak solution in $W_0^{1,p}(\Omega)$.*

Example 1.6. Let $f(x, t) = \lambda|t|^{p-2}t + \frac{(2 + \operatorname{sgn}(t))|t|^{p-2}t}{\log(1 + |t|)}$. By Proposition 2.10 below, (f_3) – (f_5) are satisfied. Therefore (1.1) has a nontrivial solution, provided $\lambda \notin \sigma(-\Delta_p)$.

Remark 1.7. (i) The condition (f_3) characterizes the problem (1.1) as asymptotically linear at infinity. In [6], Drábek and Robinson obtained a solution of (1.1) under the well known Landesman-Lazer condition. Note that they even allow $\lambda \in \sigma(-\Delta_p)$, namely, the problem is resonant at infinity. However, they have not discussed the existence of nontrivial solutions.

(ii) Theorem 1.5 improves a recent result of Guo-Liu [7, Theorem 1.2], where a nontrivial solution was obtained under the assumption (f_3) , (f_5) and (f_4') for some $\nu \in (1, p)$, there are constants $r, a > 0$ such that

$$F(x, t) \geq a|t|^\nu, \quad \text{for } (x, t) \in \Omega \times (-r, r). \quad (1.4)$$

Obviously, (f_4') is stronger than (f_4) . For the case $p = 2$, their result is due to Moroz [14, Theorem 3]. The reader may also find similar results in [12]. Note that the nonlinearity given in Example 1.6 does not satisfy the assumptions in [7, 12].

(iii) In addition to (f_3) , (f_4) and (f_5) , if $f(x, t)$ is odd in t for small $|t|$, then infinitely many solutions of the problem (1.1) were obtained by Wang [17, Theorem 2.10].

We shall prove Theorems 1.1, 1.3 and 1.5 via Morse theory. For a systematic exploration of this theory and its applications to differential equations, the reader is referred to the books by Chang [4] and Mawhin-Willem [13]. The paper is organized as follows: as preliminaries, in §2 we compute the relevant critical groups of the functional Φ and discuss the compactness issue; in §3 we prove our theorems.

2. Computation of critical groups

Let Φ be a C^1 -functional defined on a Banach space X , then the k -th critical group of Φ at an isolated critical point u with $\Phi(u) = c$ is defined by

$$C_k(\Phi, u) := H_k(\Phi_c \cap U, (\Phi_c \cap U) \setminus \{u\}), \quad k \in \mathbb{N} = \{0, 1, 2, \dots\},$$

where U is any neighborhood of u , H_* is the singular relative homology with coefficients in an Abelian group \mathcal{G} and $\Phi_c = \Phi^{-1}(-\infty, c]$.

We say that Φ satisfies the Cerami condition (C), if any sequence $\{u_n\} \subset X$ such that $\{\Phi(u_n)\}$ is bounded and $(1 + \|u_n\|) \|\Phi'(u_n)\| \rightarrow 0$ has a convergent subsequence; such a sequence is then called a Cerami sequence. If Φ satisfies the condition (C) and the critical values of Φ are bounded from below by some $\alpha > -\infty$, then the critical groups of Φ at infinity were introduced by Bartsch-Li [3] as

$$C_k(\Phi, \infty) := H_k(X, \Phi_\alpha), \quad k \in \mathbb{N}. \quad (2.1)$$

Note that by the deformation lemma, the right-hand side of (2.1) does not depend on the choice of α .

The reader is referred to [4, 13] for more details on Morse theory. In the proofs of our theorems we shall use the following result.

Proposition 2.1. *Suppose that $\Phi \in C^1(X, \mathbb{R})$ satisfies the condition (C) and Φ has only finitely many critical points, then*

- (i) *If for some $k \in \mathbb{N}$ we have $C_k(\Phi, \infty) \neq 0$, then Φ has a critical point u with $C_k(\Phi, u) \neq 0$.*
- (ii) *Let $\mathbf{0}$ be an isolated critical point of Φ . If for some $k \in \mathbb{N}$ we have $C_k(\Phi, \mathbf{0}) \neq C_k(\Phi, \infty)$, then Φ has a nonzero critical point.*

In order to apply the Morse theory to Φ , we must show that Φ satisfies the condition (C). We have the following lemma.

Lemma 2.2. *Assume that (f_*) , (f_1) and (f_2) are satisfied, then Φ satisfies the condition (C).*

Proof. This lemma has been proven in our previous paper [11]. Since that paper was written in Chinese, for the reader's convenience, we sketch the proof here briefly.

Let $\{u_n\}$ be a Cerami sequence of Φ . Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, it suffices to show that $\{u_n\}$ is bounded.

If $\{u_n\}$ is unbounded, up to a subsequence we may assume that for some $c \in \mathbb{R}$,

$$\Phi(u_n) \rightarrow c, \quad \|u_n\| \rightarrow \infty, \quad \|\Phi'(u_n)\| \|u_n\| \rightarrow 0. \quad (2.2)$$

In particular

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left(\frac{1}{p} f(x, u_n) u_n - F(x, u_n) \right) dx = \lim_{n \rightarrow \infty} \left\{ \Phi(u_n) - \frac{1}{p} \langle \Phi'(u_n), u_n \rangle \right\} = c. \quad (2.3)$$

Let $w_n = \|u_n\|^{-1} u_n$, up to a subsequence we may assume that

$$w_n \rightharpoonup w \text{ in } W_0^{1,p}(\Omega), \quad w_n \rightarrow w \text{ in } L^p(\Omega), \quad w_n(x) \rightarrow w(x) \text{ a.e. } x \in \Omega.$$

If $w = 0$, as in [8,18], we choose a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$\Phi(t_n u_n) = \max_{t \in [0,1]} \Phi(t u_n).$$

For any $m > 0$, let $v_n = (2pm)^{1/p} w_n$. Since $v_n \rightarrow 0$ in $L^q(\Omega)$ and

$$|F(x, t)| \leq C(1 + |t|^q),$$

by the continuity of the Nemitskii operator, we see that $F(\cdot, v_n) \rightarrow 0$ in $L^1(\Omega)$. Thus

$$\lim_{n \rightarrow \infty} \int_{\Omega} F(x, v_n) dx = 0.$$

So for n large enough, $(2pm)^{1/p} \|u_n\|^{-1} \in (0, 1)$, and we deduce

$$\Phi(t_n u_n) \geq \Phi(v_n) = 2m - \int_{\Omega} F(x, v_n) dx \geq m.$$

That is, $\Phi(t_n u_n) \rightarrow \infty$. Now $\Phi(\mathbf{0}) = 0$, $\Phi(u_n) \rightarrow c$, we see that $t_n \in (0, 1)$, and

$$\begin{aligned} \int_{\Omega} |\nabla(t_n u_n)|^p dx - \int_{\Omega} f(x, t_n u_n) t_n u_n dx &= \langle \Phi'(t_n u_n), t_n u_n \rangle \\ &= t_n \frac{d}{dt} \Big|_{t=t_n} \Phi(t u_n) = 0. \end{aligned}$$

Therefore, by our condition (f_2) ,

$$\begin{aligned} \int_{\Omega} \left(\frac{1}{p} f(x, u_n) u_n - F(x, u_n) \right) dx &\geq \frac{1}{\theta} \int_{\Omega} \left(\frac{1}{p} f(x, t_n u_n) t_n u_n - F(x, t_n u_n) \right) dx \\ &= \frac{1}{\theta} \left(\frac{1}{p} \int_{\Omega} |\nabla(t_n u_n)|^p dx - \int_{\Omega} F(x, t_n u_n) dx \right) \\ &= \frac{1}{\theta} \Phi(t_n u_n) \rightarrow +\infty, \end{aligned}$$

This contradicts with (2.3).

If $w \neq 0$, from the third limit in (2.2) we obtain

$$\|u_n\|^p - \int_{\Omega} f(x, u_n) u_n dx = \langle \Phi'(u_n), u_n \rangle = o(1).$$

Since $f(x, u)u \geq 0$, we deduce

$$\begin{aligned} 1 - o(1) &= \int_{\Omega} \frac{f(x, u_n) u_n}{\|u_n\|^p} dx = \left(\int_{w \neq 0} + \int_{w=0} \right) \frac{f(x, u_n) u_n}{|u_n|^p} |w_n|^p dx \\ &\geq \int_{w \neq 0} \frac{f(x, u_n) u_n}{|u_n|^p} |w_n|^p dx. \end{aligned} \tag{2.4}$$

For $x \in \Theta := \{x \in \Omega : w(x) \neq 0\}$, we have $|u_n(x)| \rightarrow +\infty$. By (1.2) we obtain

$$\frac{f(x, u_n(x)) u_n(x)}{|u_n(x)|^p} |w_n(x)|^p \rightarrow +\infty.$$

Note that the Lebesgue measure of Θ is positive, using the Fatou Lemma we deduce

$$\int_{w \neq 0} \frac{f(x, u_n)u_n}{|u_n|^p} |w_n|^p dx \rightarrow +\infty.$$

This contradicts with (2.4).

Therefore, $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. \square

Remark 2.3. The argument that we employed to exclude the case $w = 0$, is due to Jeanjean [8] and Zou [18]. They used this technique to show that for every potential critical value c , there exists a bounded $(PS)_c$ sequence. While Lemma 2.2 indicates that, although there may exist unbounded (PS) sequence, all Cerami sequences are bounded.

For the proofs of our theorems, in what follows we may assume that Φ has only finitely many critical points. Since Φ satisfies the condition (C), the critical groups $C_*(\Phi, \infty)$ of Φ at infinity make sense. To simplify the notations we denote $X = W_0^{1,p}(\Omega)$.

Lemma 2.4. *Assume that (f_*) , (f_1) and (f_2) are satisfied, then $C_k(\Phi, \infty) \cong 0$ for all $k \in \mathbb{N}$.*

Proof. Let $S = \{u \in X : \|u\| = 1\}$. By (1.2) it is easy to see that for any $u \in S$, we have

$$\Phi(tu) \rightarrow -\infty, \quad \text{as } t \rightarrow +\infty.$$

Choose

$$a < \min \left\{ \inf_{\|u\| \leq 1} \Phi(u), 0 \right\}.$$

Then for any $u \in S$, there exists $t_0 > 1$ such that $\Phi(t_0u) \leq a$. By (f_2) , we have

$$\mathcal{F}(x, z) \geq 0, \quad \text{for } (x, z) \in \Omega \times \mathbb{R}. \quad (2.5)$$

Therefore, if

$$\Phi(tu) = \frac{t^p}{p} - \int_{\Omega} F(x, tu) dx \leq a,$$

using (2.5) we obtain

$$\begin{aligned} \frac{d}{dt} \Phi(tu) &= t^{p-1} - \int_{\Omega} u f(x, tu) dx \\ &\leq \frac{1}{t} \left\{ pa + \int_{\Omega} pF(x, tu) dx - \int_{\Omega} tu f(x, tu) dx \right\} \\ &= \frac{1}{t} \left\{ pa - \int_{\Omega} \mathcal{F}(x, tu) dx \right\} \\ &\leq \frac{1}{t} pa < 0. \end{aligned}$$

Therefore, by the Implicit Function Theorem, there exists a unique $T \in C(S, \mathbb{R})$ such that $\Phi(T(u)u) = a$.

Using the function T , we can follow the argument in [10, Page 4] to construct a strong deformation retract from $X \setminus \{0\}$ to Φ_a , and deduce

$$C_k(\Phi, \infty) = H_k(X, \Phi_a) \cong H_k(X, X \setminus \{0\}) = 0. \quad \square$$

Remark 2.5. Result similar to Lemma 2.4 has been obtained (for $p = 2$) by Wang [16], under the condition (f_*) and (AR). This result was then generalized to general $p > 1$ by Liu [10].

Lemma 2.6. *Assume that (f_*) , (f_4) and (f_5) are satisfied, then $C_k(\Phi, 0) \cong 0$ for all $k \in \mathbb{N}$.*

Proof. For $u \in X \setminus \{0\}$, we have

$$\lim_{|s| \rightarrow 0} \frac{\Phi(su)}{|s|^p} = -\infty. \quad (2.6)$$

In fact, for any $\{s_n\} \subset \mathbb{R}$ with $s_n \rightarrow 0$, let $v_n = s_n u$. Then

$$v_n \rightarrow 0 \text{ in } W_0^{1,p}(\Omega), \quad v_n(x) \rightarrow 0 \text{ a.e. } x \in \Omega.$$

From (1.3), by the Fatou Lemma we have

$$\liminf_{n \rightarrow \infty} \int_{\Omega} \frac{F(x, v_n)}{|v_n|^p} |u|^p \, dx \geq \int_{\Omega} \liminf_{n \rightarrow \infty} \frac{F(x, v_n)}{|v_n|^p} |u|^p \, dx = +\infty.$$

Thus

$$\begin{aligned} \frac{\Phi(s_n u)}{|s_n|^p} &= \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} \frac{F(x, s_n u)}{|s_n|^p} \, dx \\ &= \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} \frac{F(x, v_n)}{|v_n|^p} |u|^p \, dx \rightarrow -\infty, \end{aligned}$$

and (2.6) follows.

Using (2.6), we see that for any $u \in X \setminus \{0\}$, there exists $s_0 \in (0, 1)$ such that

$$\Phi(su) < 0, \quad \text{for } s \in (0, s_0). \quad (2.7)$$

Assume that $\Phi(u) \geq 0$, that is

$$\int_{\Omega} |\nabla u|^p \, dx \geq p \int_{\Omega} F(x, u) \, dx.$$

Then according to (f₅), for $u \in \Phi^{-1}[0, +\infty) \setminus \{0\}$ we obtain

$$\begin{aligned} \frac{d}{ds} \Big|_{s=1} \Phi(su) &= \frac{d}{ds} \Big|_{s=1} \left(\frac{s^p}{p} \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} F(x, su) \, dx \right) \\ &= \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} f(x, u)u \, dx \\ &\geq \int_{\Omega} (pF(x, u) - f(x, u)u) \, dx > 0. \end{aligned} \quad (2.8)$$

Now we adjust the argument in the proof of [9, Proposition 2.1] slightly. From (2.7) and (2.8), we see that for any $u \in \Phi^{-1}[0, +\infty) \setminus \{0\}$, there exists a unique $T = T(u) > 0$ such that $\Phi(Tu) = 0$. Moreover, since $\Phi(Tu) = 0$, from (2.8) we deduce

$$\frac{d}{dt} \Big|_{t=T} \Phi(tu) = \frac{1}{T} \frac{d}{ds} \Big|_{s=1} \Phi(sTu) > 0.$$

Thus, by the Implicit Function Theorem we see that T is continuous on $\Phi^{-1}[0, +\infty) \setminus \{0\}$. If $\Phi(u) \leq 0$, we set $T(u) = 1$. Then $T : X \rightarrow \mathbb{R}$ is continuous.

We now define $\eta : [0, 1] \times X \rightarrow X$ by

$$\eta(s, u) = (1 - s)u + sT(u)u, \quad (s, u) \in [0, 1] \times X.$$

It is easy to see that η is a continuous deformation from $(X, X \setminus \{0\})$ to $(\Phi_0, \Phi_0 \setminus \{0\})$, hence

$$C_k(\Phi, 0) = H_k(\Phi_0, \Phi_0 \setminus \{0\}) \cong H_k(X, X \setminus \{0\}) = 0, \quad k \in \mathbb{N}. \quad \square$$

Remark 2.7. (i) For related results, see [7,9,12,14]. Note that the assumptions in all these papers imply (1.4). Our Lemma 2.6 improves [7, Theorem 2.1] and [14, Theorem 1].

- (ii) To obtain the local result described in Lemma 2.6, we have to assume the global condition (f_5) . This unpleasant phenomenon has appeared in [14, Theorem 1], and has been discussed by Moroz [14, Remark 1.3]. As in [14], if $N = 1$ and Ω is an open interval in \mathbb{R} , since there is a continuous embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega), \quad (2.9)$$

we can also replace (f_5) with a local condition (f'_5) , see the following Corollary 2.8.

Corollary 2.8. *Let $N = 1$ and Ω be an open interval in \mathbb{R} . Assume that (f_\star) , (f_4) and (f'_5) there exists $\rho > 0$ such that $pF(x, t) - f(x, t)t > 0$ for $x \in \Omega$ and $0 < |t| < \rho$, are satisfied, then $C_k(\Phi, \mathbf{0}) = 0$ for all $k \in \mathbb{N}$.*

Proof. Thanks to the embedding (2.9), using (f'_5) we can choose some $\delta > 0$ such that (2.8) is still valid provided $\Phi(u) \geq 0$ and $0 < \|u\| \leq \delta$. Then the argument in the proof of [9, Proposition 2.1] yields the desired result. \square

Remark 2.9. This corollary improves Moroz [14, Theorem 4]. In the case $N = 1$, Corollary 2.8 also improves the result obtained by Jiu-Su [9, Proposition 2.1], where they assumed (f_0^1) there exist $\rho > 0$ and $\mu \in (0, p)$ such that

$$\mu F(x, t) \geq f(x, t)t > 0, \quad \text{for } x \in \Omega \text{ and } 0 < |t| < \rho.$$

This condition may be viewed as a dual version of (AR). Obviously, (f_0^1) is stronger than (f_4) and (f'_5) .

To conclude this section, we prove the following proposition.

Proposition 2.10. *If for fixed $x \in \Omega$, $\frac{f(x, t)}{|t|^{p-2}t}$ is strictly decreasing in $(0, +\infty)$ and strictly increasing in $(-\infty, 0)$, then (f_5) is satisfied.*

Proof. We consider the case $t > 0$. For $s \in (0, t)$ we have

$$f(x, s) > \frac{f(x, t)}{|t|^{p-2}t} |s|^{p-2} s.$$

Integrating this inequality over $(0, t)$, we deduce

$$F(x, t) = \int_0^t f(x, s) ds > \frac{f(x, t)}{|t|^{p-2}t} \int_0^t |s|^{p-2} s ds = \frac{1}{p} f(x, t)t.$$

The case that $t < 0$ is similar. \square

3. The proofs of the theorems

In this section we prove our theorems.

Proof of Theorem 1.1. Since $f(x, 0) = 0$, the zero function $\mathbf{0}$ is a trivial critical point of Φ . By [10, Lemma 2.3], the condition (f_0) implies that $C_1(\Phi, \mathbf{0}) \neq 0$. While according to Lemma 2.4, $C_1(\Phi, \infty) = 0$. Now the desired result follows from Proposition 2.1. \square

To prove Theorem 1.3, we need the following results of Perera [15]. Let

$$I_\lambda = \frac{1}{p} \int_\Omega (|\nabla u|^p - \lambda |u|^p) dx, \quad u \in X = W_0^{1,p}(\Omega).$$

Proposition 3.1 ([15, Lemma 4.1]). *Assume that (f_*) and (f_0^*) hold, then there are $\rho > 0$ and $\tilde{\Phi} \in C^1(X, \mathbb{R})$ such that*

$$\tilde{\Phi}(u) = \begin{cases} I_\lambda(u), & \|u\| \leq \rho, \\ \Phi(u), & \|u\| \geq 2\rho; \end{cases}$$

and $\mathbf{0}$ is the only critical point of Φ and $\tilde{\Phi}$ with $\|u\| \leq 2\rho$.

Remark 3.2. Checking the proof of [15, Lemma 4.1] carefully, we find that

$$\varepsilon := \inf_{\rho \leq \|u\| \leq 2\rho} \|\tilde{\Phi}'(u)\| > 0.$$

Therefore, it is easy to see that if Φ satisfies the condition (C), so does $\tilde{\Phi}$.

Now we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3. Since $\lambda \notin \sigma(-\Delta_p)$, we may assume that for some $\ell \in \mathbb{N}$,

$$\lambda \in (\lambda_\ell, \lambda_{\ell+1}) \setminus \sigma(-\Delta_p);$$

where $\lambda_0 = -\infty$ and $\{\lambda_\ell\}_{\ell=1}^\infty$ is a sequence of eigenvalues recently constructed by Perera [15]. By [15, Proposition 1.1] we see that

$$C_\ell(I_\lambda, \mathbf{0}) \neq 0.$$

Take $\tilde{\Phi}$ as described in Proposition 3.1. We first note that by Remark 3.2, $\tilde{\Phi}$ satisfies the condition (C). Since $I_\lambda = \tilde{\Phi}$ in a neighborhood of $\mathbf{0}$, we have

$$C_\ell(\tilde{\Phi}, \mathbf{0}) = C_\ell(I_\lambda, \mathbf{0}) \neq 0. \quad (3.1)$$

On the other hand, since $\tilde{\Phi} = \Phi$ outside a big ball B , if we choose

$$\alpha < \min \left\{ \inf_B \tilde{\Phi}, \inf_B \Phi \right\}$$

in the definition of critical groups at infinity (2.1), we see that

$$C_\ell(\tilde{\Phi}, \infty) = C_\ell(\Phi, \infty) = 0, \quad (3.2)$$

according to Lemma 2.4. Comparing (3.1) and (3.2), by Proposition 2.1, we deduce that $\tilde{\Phi}$ has a nonzero critical point u . By Proposition 3.1, u is also a nonzero critical point of Φ . \square

Proof of Theorem 1.5. The proof is ‘dual’ to the proof of Theorem 1.3. Since we have obtained $C_k(\Phi, \mathbf{0}) = 0$ for all $k \in \mathbb{N}$ in Lemma 2.6, we may proceed as in the proof of [12, Theorem 1.2]. We omit the details. \square

Using Corollary 2.8, for the case $N = 1$ and Ω is an open interval in \mathbb{R} , in Theorem 1.5 we may replace the global assumption (f_5) with the local one (f'_5) .

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