Remarks on Multiple Solutions for Elliptic Resonant Problems ☆

Shibo Liu [a]

[a] *Department of Mathematics, Xiamen University, Xiamen 361005, China*

Abstract

We obtain four nontrivial solutions for an elliptic resonant problem via Morse theory and Lyapunov-Schmidt reduction method. Our result improves some recent works.

1. Introduction

Let Ω be a smooth bounded region in $\mathbb{R}^N$, and $-\Delta$ the Laplacian operator, $p : \mathbb{R} \to \mathbb{R}$ be a differentiable function, we consider the boundary value problem

$$\begin{cases} \Delta u + p(u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Let $\lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \cdots$ be the eigenvalues of $-\Delta$ with Dirichlet boundary condition in Ω. We assume that p satisfies the following conditions:

(p_1) $p \in C^1(\mathbb{R}, \mathbb{R})$, $p(0) = 0$, $p'(0) < \lambda_1 < p_\infty = \lambda_m$, where

$$p_\infty = \lim_{|t|\to\infty} \frac{p(t)}{t} \in \mathbb{R}, \tag{1.2}$$

(p_2) For some $\gamma \in \mathbb{R}$, $p'(t) \le \gamma < \lambda_{m+1}$.

The condition (1.2) means that our problem (1.1) is 'asymptotically linear' at infinity. Since the pioneer work of Amann-Zehnder [2], this kind of problems have captured great interest. It is well known that, if the nonlinearity crosses at least one eigenvalue, then in general (1.1) has a nontrivial solution. In the case that $p'(0) < \lambda_1$, one can even obtain three nontrivial solutions, see for instance [8].

The existence of more solutions requires more conditions on the nonlinearity. Therefore, in what follows, in addition to (p_1) we assume (p_2). Then, the existence of four nontrivial solutions of (1.1) was obtained by Castro-Cossio [6], provided $p_\infty \in (\lambda_m, \lambda_{m+1})$ is not an eigenvalue. Recently, Li-Zhang [10] was able to deal with the resonant case: $p_\infty = \lambda_m$. They assumed (p_1), (p_2) and

☆ Supported by National Natural Science Foundation of China (10601041).
Email address: `liusb@stu.edu.cn` (Shibo Liu).

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(p_3) There exists $\alpha \in [0, 1)$ and $c > 0$ such that

(p_3^1) $|p(t) - \lambda_m t| \le c(1 + |t|^\alpha)$,

(p_3^2) $(P(t) - \frac{1}{2}\lambda_m t^2)\,|t|^{-2\alpha} \to +\infty$, as $|t| \to \infty$

and showed that (1.1) has four nontrivial solutions, where $P(t) = \int_0^t p(s)\,\mathrm{d}s$.

More recently, Liu-Li [11] also obtained four nontrivial solutions for the resonant case $p_\infty = \lambda_m$, assuming (p_1), (p_2) and

(p_4) $(P(t) - \frac{1}{2}\lambda_m t^2)\,|t|^{-1} \to +\infty$, as $|t| \to \infty$.

This result does not require (p_3^1) and, if $\alpha > \frac{1}{2}$, (p_4) is weaker than (p_3^2).

In this note, we shall improve the last result further. In fact, we can prove the following theorem, which generalizes the results of [10] and [11] mentioned above.

Theorem 1.1. *Assume that* (p_1), (p_2) *and*

(p_5) $P(t) - \frac{1}{2}\lambda_m t^2 \to +\infty$, *as* $|t| \to \infty$,

then the problem (1.1) has at least four nontrivial solutions.

Our proof of this theorem is variational. Let $H_0^1(\Omega)$ be the Sobolev space with inner product

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v\,\mathrm{d}x$$

and corresponding norm $\|u\| = \langle u, u\rangle^{1/2}$, we shall prove the theorem by showing that, the C^2-functional $f : H_0^1(\Omega) \to \mathsf{R}$,

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega P(u)\,\mathrm{d}x \tag{1.3}$$

possess four nonzero critical points. To do that, we shall apply the Morse theory (see the monographs [7, 13] for a systematic exploration).

However, unlike the above mentioned references, under our assumptions, the functional f may *not* satisfy the Palais-Smale (PS) condition. Thus we can not apply the Morse theory to f directly. To go around this difficulty, we first observe that, although f may not satisfy (PS), the truncated functional $f_\pm$ does satisfy the (PS) condition. This observation enables us to obtain two critical points of f via the mountain pass lemma. Then, thanks to condition (p_2), we can apply the Lyapunov-Schmidt reduction procedure and turn to consider a reduced functional φ, which is defined on a finite dimensional space. As observed by Liu-Tang-Wu [12], it turns out that φ is anti-coercive, hence satisfies (PS). We will eventually apply Morse theory to φ to produce the last two nonzero critical points of f. To carry out this program, we will investigate the relation between the critical groups of f and φ at corresponding critical points in Section 2. The proof of Theorem 1.1 will be given in Section 3.

2. Critical groups and Lyapunov-Schmidt reduction

Let X be a Banach space, $f \in C^1(X, \mathsf{R})$ and u be an isolated critical point of f with $f(u) = c$. Then the groups

$$C_q(f, u) := H_q(f_c, f_c \setminus \{u\}), \qquad q = 0, 1, 2, \cdots$$

is called the q^{th} *critical group* of f at u, here $f_c = f^{-1}(-\infty, c]$, $H_q(A, B)$ stands for the q^{th} singular relative homology group of the topological pair (A, B) with coefficients in an Abelian group $\mathcal{G}$.

Assume that f satisfies (PS), f has no critical value less than $\alpha \in \mathbb{R}$, then the q^{th} *critical group at infinity* of f is defined in [4, Definition 3.4] as

$$C_q(f, \infty) := H_q(X, f_\alpha), \qquad q = 0, 1, 2, \cdots.$$

Note that these groups are not dependent on the choice of α.

The critical groups of f at an isolated critical point u describe the local behavior of f near u, while the critical groups of f at infinity describe the global property of f. The Morse inequality gives the relation between them.

Proposition 2.1 ([7, Theorem I.4.3]). *Suppose that $f \in C^1(X, \mathbb{R})$ satisfies (PS), has only isolated critical points, and the critical values of f are bounded below. Then we have*

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1 + t)Q(t), \tag{2.1}$$

where $M_q = \sum_{f'(u)=0} \operatorname{rank} C_q(f, u)$, $\beta_q = \operatorname{rank} C_q(f, \infty)$, Q is a formal series with nonnegative integer coefficients.

Next, we recall the Lyapunov-Schmidt reduction method. For the proof, the reader is referred to [1, 5].

Proposition 2.2. *Let X be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, X^-, X^+ be closed subspaces of X such that $X = X^- \oplus X^+$. Let $f : X \to \mathbb{R}$ be C^1-functional. If there is a real number $\beta > 0$ such that for all $v \in X^-$ and $w_1, w_2 \in X^+$, there holds*

$$\langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \beta \|w_1 - w_2\|^2, \tag{2.2}$$

then we have

(i) *There exists a continuous function $\psi : X^- \to X^+$ such that*

$$f(v + \psi(v)) = \min_{w \in X^+} f(v + w).$$

Moreover, $\psi(v)$ is the unique member of X^+ such that

$$\langle \nabla f(v + \psi(v)), w \rangle = 0, \qquad \text{for all } w \in X^+.$$

(ii) *The functional $\varphi : X^- \to \mathbb{R}$ defined by $\varphi(v) = f(v + \psi(v))$ is of class C^1, and*

$$\langle \nabla \varphi(v), v_1 \rangle = \langle \nabla f(v + \psi(v)), v_1 \rangle, \qquad \text{for all } v, v_1 \in X^-.$$

(iii) *An element $v \in X^-$ is a critical point of φ if and only if $v + \psi(v)$ is a critical point of f.*

According to Proposition 2.2, if v is an isolated critical point of φ, then $v + \psi(v)$ is an isolated critical point of f, and vice versa. One may ask: is there any relation between the critical groups $C_*(\varphi, v)$ and $C_*(f, v + \psi(v))$? The following lemma is devoted to this issue.

Lemma 2.3. *Assume that the assumptions of Proposition 2.2 hold, then at any isolated critical point v of φ we have*

$$C_q(\varphi, v) \cong C_q(f, v + \psi(v)), \qquad q = 0, 1, 2, \cdots.$$

Proof. Let $c = \varphi(v) = f(v + \psi(v))$,

$$A = \{(v, \psi(v)) : \ v \in \varphi_c\}.$$

Then we have the following maps between topological pairs

$$(f_c, f_c \backslash \{u\}) \xrightarrow{h} (A, A \backslash \{u\}) \xleftarrow{g} (\varphi_c, \varphi_c \backslash \{v\}), \tag{2.3}$$

where $u = v + \psi(v)$, $h(v, w) = (v, \psi(v))$ and $g(v) = (v, \psi(v))$.

Obviously, g is a homeomorphism with inverse $g^{-1}(v, \psi(v)) = v$. On the other hand, (2.2) implies that f is convex on w, that is, for $v \in X^-$ and $w_1, w_2 \in X^+$,

$$f(v, (1-t)w_1 + tw_2) \leq (1-t)f(v, w_1) + tf(v, w_2), \qquad 0 \leq t \leq 1.$$

Hence we can define $F : ([0, 1] \times f_c, [0, 1] \times (f_c \backslash \{u\})) \to (f_c, f_c \backslash \{u\})$,

$$F(t, (v, w)) = (v, (1-t)w + t\psi(v)).$$

Using the homotopy F, let $i : (A, A \backslash \{u\}) \to (f_c, f_c \backslash \{u\})$ be the inclusion, it is easy to see that $i \circ h \simeq \mathbf{1}_{(f_c, f_c \backslash \{u\})}$, $h \circ i = \mathbf{1}_{(A, A \backslash \{u\})}$. So h is a homotopic equivalence.

Now passing to homology in (2.3), one sees that h_* and g_* are isomorphisms. Thus we obtain

$$C_q(f, v + \psi(v)) = H_q(f_c, f_c \backslash \{u\}) \cong H_q(\varphi_c, \varphi_c \backslash \{v\}) = C_q(\varphi, v).$$

This completes the proof. □

Now assume that there is a compact mapping $T : X \to X$ such that for any $u \in X$,

$$\nabla f(u) = u - T(u). \tag{2.4}$$

Since $\nabla\varphi(v)$ is exactly the orthogonal projection of $\nabla f(v + \psi(v))$ in X^-, $\nabla\varphi(v)$ is also of the form (2.4). Therefore the Leray-Schauder indices $\mathrm{ind}(\nabla\varphi, v)$ and $\mathrm{ind}(\nabla f, v + \psi(v))$ are well defined, provided v is an isolated critical point of φ.

Corollary 2.4. *Under the assumptions of Proposition 2.2, if there exists a compact mapping $T : X \to X$ such that (2.4) holds, then we have*

$$\mathrm{ind}(\nabla\varphi, v) = \mathrm{ind}(\nabla f, v + \psi(v))$$

at any isolated critical point v of φ.

Proof. By Lemma 2.3 and [7, Theorem II.3.2], we have

$$\begin{aligned} \mathrm{ind}(\nabla\varphi, v) &= \sum_{q=0}^{\infty} (-1)^q \mathrm{rank}\, C_q(\varphi, v) \\ &= \sum_{q=0}^{\infty} (-1)^q \mathrm{rank}\, C_q(f, v + \psi(v)) \\ &= \mathrm{ind}(\nabla f, v + \psi(v)). \end{aligned}$$

□

Remark 2.5. For related results, see [6, Lemma 2.1] and [9, Lemma 2.6].

3. The proof of Theorem 1.1

Now we can give the proof of Theorem 1.1. As the first step, we consider the truncated problem

$$\begin{cases} \Delta u + p_+(u) = 0, \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where

$$p_+(t) = \begin{cases} p(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

We observe that

$$\lim_{t \to -\infty} \frac{p_+(t)}{t} = 0, \qquad \lim_{t \to +\infty} \frac{p_+(t)}{t} = p_\infty,$$

that is, the nonlinearity p_+ in the problem (3.1) is asymptotic to $(0, p_\infty)$. Since $p_\infty > \lambda_1$, the point $(0, p_\infty)$ is not contained in the Fučik spectrum of the Laplacian operator $-\Delta$. Hence, the variational functional $f_+ : H_0^1(\Omega) \to \mathsf{R}$ of (3.1), given by

$$f_+(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x - \int_\Omega P_+(u) \, \mathrm{d}x,$$

satisfies the (PS) condition, where $P_+(t) = \int_0^t p_+(s)ds$. See [8, pp. 181–182] for the detail.

It is easy to see that the zero function $\mathbf{0}$ is a local minimizer of f_+, and $f_+(t\phi_1) \to -\infty$ as $t \to +\infty$, where $\phi_1 > 0$ is a first eigenfunction of $(-\Delta, H_0^1(\Omega))$. Thus, by the mountain pass lemma we obtain a critical point u_+ of f_+. By the standard argument as in [7, 8], we deduce that u_+ is a critical point of f, with

$$C_q(f, u_+) \cong \delta_{q,1}\mathcal{G}, \qquad u_+ > 0 \text{ in } \Omega. \tag{3.2}$$

Similarly, we obtain another critical point u_- of f, such that

$$C_q(f, u_-) \cong \delta_{q,1}\mathcal{G}, \qquad u_- < 0 \text{ in } \Omega. \tag{3.3}$$

In what follows, we will prove that f has two more nonzero critical points. We decompose $H_0^1(\Omega) = X^- \oplus X^+$ according to $p_\infty = \lambda_m$. More precisely, we set

$$X^- = \bigoplus_{i=1}^{m} \ker(-\Delta - \lambda_i), \qquad X^+ = (X^-)^\perp = \overline{\bigoplus_{i \geq m+1} \ker(-\Delta - \lambda_i)}.$$

Since $p'(t) \leq \gamma < \lambda_{m+1}$, for any $v \in X^-$ and $w_1, w_2 \in X^+$ we have

$$\langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \beta \|w_1 - w_2\|^2,$$

where $\beta = 1 - \gamma\lambda_{m+1}^{-1}$. The details can be found in [6, Section 2].

Thus, by Proposition 2.2, we have a continuous map $\psi : X^- \to X^+$ and a C^1-function $\varphi : X^- \to \mathsf{R}$, such that

$$\varphi(v) = f(v + \psi(v)) = \min_{w \in X^+} f(v + w). \tag{3.4}$$

We need to show that φ has at least five critical points. Hence in what follows, we assume that for some $\alpha \in \mathsf{R}$, φ has no critical value less than α.

Although we can not prove that f satisfies (PS), we will show in the next lemma that φ is anti-coercive. Then, noting that $\dim X^- < \infty$, we see that φ satisfies (PS).

Lemma 3.1. *Assume that $p \in C^1(\mathsf{R}, \mathsf{R})$ satisfies (p_1), (p_2) and (p_5), then the functional φ given in (3.4) is anti-coercive.*

Proof. This lemma has been proved by Liu-Tang-Wu [12, Lemma 2]. Their proof is a little complicated and involves an argument in the proof of [3, Lemma 3.2]. Here we give a different proof, which is self-contained, and considerably simpler.

According to (p_5), there exists $R > 0$, such that

$$\frac{1}{2}\lambda_m t^2 - P(t) \le 0, \qquad \text{if } |t| \ge R.$$

Therefore, for any $t \in \mathbb{R}$, we have

$$\frac{1}{2}\lambda_m t^2 - P(t) \le M := \max_{|t|\le R}\left|\frac{1}{2}\lambda_m t^2 - P(t)\right|. \tag{3.5}$$

Assume now $\{v_n\}_{n=1}^{\infty}$ is a sequence in X^- such that $\|v_n\| \to \infty$. Let $z_n = v_n/\|v_n\|$. Then $\|z_n\| = 1$. Since $\dim X^- < \infty$, there is some $z \in X^-$ such that up to a subsequence, $\|z_n - z\| \to 0$. Hence $\|z\| = 1$. In particular, $z \ne \mathbf{0}$, the set

$$\Theta = \{x \in \Omega : \ z(x) \ne 0\}$$

is of positive measure. For $x \in \Theta$, we have $|v_n(x)| \to \infty$. Hence by (p_5) and the Fatou Lemma,

$$\int_\Theta \left(\frac{1}{2}\lambda_m v_n^2 - P(v_n)\right) \mathrm{d}x \to -\infty, \qquad \text{as } n \to \infty. \tag{3.6}$$

By (3.5) and (3.6), we have

$$\begin{aligned}
\varphi(v_n) \le f(v_n) &= \frac{1}{2}\int_\Omega |\nabla v_n|^2 - \int_\Omega P(v_n)\,\mathrm{d}x \\
&\le \int_\Omega \left(\frac{1}{2}\lambda_m v_n^2 - P(v_n)\right)\mathrm{d}x \\
&\le \int_\Theta \left(\frac{1}{2}\lambda_m v_n^2 - P(v_n)\right)\mathrm{d}x + \int_{\Omega\setminus\Theta}\left(\frac{1}{2}\lambda_m v_n^2 - P(v_n)\right)\mathrm{d}x \\
&\le \int_\Theta \left(\frac{1}{2}\lambda_m v_n^2 - P(v_n)\right)\mathrm{d}x + M\cdot \operatorname{meas}(\Omega\setminus\Theta) \to -\infty.
\end{aligned}$$

This concludes the proof. □

Having verified that φ is anti-coercive, we can follow the argument in the proof of [11, Lemma 2.1] to compute $C_*(\varphi,\infty)$. For the reader's convenience, we describe it briefly.

Since φ is anti-coercive, we can choose $a < b < \alpha$ and $\rho > r > 0$ such that

$$A_\rho \subset \varphi_a \subset A_r \subset \varphi_b,$$

where $A_\rho = \{v \in X^- : \ \|v\| \ge \rho\}$. Note that $H_*(\varphi_b, \varphi_a) = 0$, because φ has no critical value in $[a, b]$. Thus we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 = H_q(\varphi_b,\varphi_a) & \longrightarrow & H_q(X^-,\varphi_a) & \xrightarrow{i_*} & H_q(X^-,\varphi_b) & \xrightarrow{\partial_*} & H_{q-1}(\varphi_b,\varphi_a) = 0 \\
\uparrow & & \uparrow & \searrow^{\ell_*} & \uparrow & & \uparrow \\
0 = H_q(A_r,A_\rho) & \longrightarrow & H_q(X^-,A_\rho) & \xrightarrow{k_*} & H_q(X^-,A_r) & \xrightarrow{\partial_*} & H_{q-1}(A_r,A_\rho) = 0
\end{array}$$

where all the homomorphisms except ∂_* are induced by inclusions. The exactness of the rows implies that i_* and k_* are isomorphisms. Hence ℓ_* is also an isomorphism, and we get

$$C_q(\varphi,\infty) = H_q(X^-,\varphi_\alpha) \cong H_q(X^-,A_r) = \delta_{q,m}\mathcal{G}.$$

Because the anti-coercive functional φ is defined on the m-dimensional space X^-, it has a maximal point v, with

$$C_q(\varphi, v) \cong \delta_{q,m}\mathcal{G}.$$

Let $\boldsymbol{\theta}$, v_+ and v_- be the projection of $\mathbf{0}$, u_+ and u_- in X^- respectively. Then they are all critical points of φ. By (3.2), (3.3) and Lemma 2.3, and remember that $\mathbf{0}$ is a local minimizer of f, we have

$$C_q(\varphi, v_\pm) \cong C_q(f, u_\pm) \cong \delta_{q,1}\mathcal{G},$$
$$C_q(\varphi, \boldsymbol{\theta}) \cong C_q(f, \mathbf{0}) \cong \delta_{q,0}\mathcal{G}.$$

If $\boldsymbol{\theta}$, v_+, v_- and v are the only critical points of φ, then the Morse inequality (2.1) with $t = -1$ becomes

$$(-1)^0 + 2 \times (-1)^1 + (-1)^m = (-1)^m.$$

This is impossible. Thus φ has at least five critical points. So f also has five critical points, four of which are nonzero. Therefore the problem (1.1) has four nontrivial solutions. This completes the proof of Theorem 1.1.

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