Multiple Solutions for Coercive $p$-Laplacian Equations

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Abstract

We obtain multiple nonzero solutions for coercive $p$-Laplacian equations. In order to obtain the third nonzero solution, we use the second deformation lemma to construct the desired mountain pass path.

Key words: $p$-Laplacian, minimizers, mountain pass lemma, the second deformation lemmas

1. Introduction

In this paper, we consider the existence of multiple nonzero solutions of the Dirichlet boundary value problem

\[\begin{cases}
-\Delta_p u = f(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega.
\end{cases}\]  \hspace{1cm} (1.1)

Here $p > 1$, $-\Delta_p u := -\text{div}(\nabla u^{p-2} \nabla u)$ denotes the $p$-Laplacian operator, $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

We assume that the nonlinearity $f(x, u)$ satisfies the following condition

\((f_*)\) \quad f : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function with subcritical growth}

\[|f(x, t)| \leq C_1 (1 + |t|^\theta - 1), \quad \text{for a.e. } x \in \Omega, t \in \mathbb{R},\]

where $C_1 > 0$, $\theta \in (p, p^*)$ and $p^* = Np/(N - p)$ for $p < N$, while $p^* = +\infty$ if $p \geq N$.

Then, it is well known that the weak solutions of the problem (1.1) are exactly the critical points of the $C^1$-functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$,

\[\Phi(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega F(x, u) \, dx,\]  \hspace{1cm} (1.2)

where $F(x, t) = \int_0^t f(x, s) \, ds$, and $W_0^{1,p}(\Omega)$ is the well-known Sobolev space endowed with the norm $\|u\| = (\int_\Omega |\nabla u|^p \, dx)^{1/p}$. For $q \geq 1$, we denote by $|u|_q = (\int_\Omega |u|^q \, dx)^{1/q}$ the usual Lebesgue norm on $L^q(\Omega)$.

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Let $\lambda_1$ and $\lambda_2$ be the first and the second eigenvalues of $-\Delta_p$ on $W^{1,p}_0(\Omega)$. It is well known that $\lambda_1 > 0$ is a simple eigenvalue with fixed-sign eigenfunctions. By $\phi_1$ we denote the positive eigenfunction of $\lambda_1$ such that $|\phi_1|_p = 1$. We also know that $\sigma(-\Delta_p) \cap (\lambda_1, \lambda_2) = \emptyset$, where $\sigma(-\Delta_p)$ is the spectrum of $-\Delta_p$, see [9].

In this paper, we will study the existence of multiple solutions of the problem (1.1) in the case that $\Phi$ is coercive. More precisely, we have the following result.

**Theorem 1.1.** Assume that $(f_*)$ holds. If in addition we have

\begin{equation}
\begin{aligned}
(f_0^1) \quad &\text{there exists } \rho > 0 \text{ such that } F(x,t) \geq \frac{\lambda_1}{p} |t|^p \text{ for } |t| \leq \rho \text{ and } x \in \Omega, \\
(f_\infty) \quad \limsup_{|t| \to \infty} \frac{pF(x,t)}{|t|^p} < \lambda_1,
\end{aligned}
\end{equation}

then the problem (1.1) has at least two nonzero solutions.

Recall that assuming $(f_*)$, $(f_\infty)$ and

\begin{equation}
\begin{aligned}
(f_0^*) \quad &\text{there exist } \rho > 0 \text{ and } \lambda \in (\lambda_1, \lambda_2) \text{ such that for } |t| \leq \rho \text{ and } x \in \Omega \text{ there holds}
\end{aligned}
\end{equation}

\[\lambda_1 |t|^p \leq pF(x,t) \leq \lambda |t|^p,
\]

the existence of two nonzero solutions of the problem (1.1) has been obtained by Liu-Su [13, Theorem 1.1]. Here, our condition $(f_0^1)$ is much weaker than their $(f_0^*)$. In their argument, the condition $(f_0^*)$ is used to produce a local linking at zero, then the famous three critical points theorem [3, 11] is applied. Our proof of Theorem 1.1 is completely different. We will use the truncated method to produce a nonpositive solution and a nonnegative one for our problem (1.1).

If we impose stronger conditions on the nonlinearity $f(x,u)$, we can obtain one more solution.

**Theorem 1.2.** Assume that $(f_*)$ and $(f_\infty)$ hold, $f(x,t)t \geq 0$ for $x \in \Omega$ and $t \in \mathbb{R}$. If in addition we have

\begin{equation}
\begin{aligned}
(f_0^2) \quad &\text{there exist } \rho > 0 \text{ and } \lambda > \lambda_2 \text{ such that } F(x,t) \geq \frac{\lambda}{p} |t|^p \text{ for } |t| \leq \rho
\end{aligned}
\end{equation}

then the problem (1.1) has at least three nonzero solutions.

We refer to [6, 10, 17] for recent work on the existence of three nonzero solutions for coercive elliptic equations. In all these papers the asymptotic limits

\begin{equation}
\begin{aligned}
a_{\pm} = \lim_{t \to 0^\pm} \frac{f(x,t)}{|t|^{p-2}t}
\end{aligned}
\end{equation}

play an essential role. Here we do not require that the limits exist. For the case that $f$ is ‘superlinear’ at zero, that is, $F(x,t) \geq a|t|^\mu$ for small $|t|$ and some $\mu \in (1, p)$, see [14] for $p = 2$ and [12] for general $p > 1$. Obviously, this ‘superlinear’ condition is stronger than $(f_0^2)$.

Now let us say a few words about the proof of Theorem 1.2. It turns out that the two nonzero solutions obtained in Theorem 1.1 are local minimizers of the functional $\Phi$. Naturally, the third solution will be found via the mountain pass lemma [1]. However, we have to show that this solution is not the trivial one, i.e., the zero function $0$. To this end, we will construct a curve $\gamma$ between the two local minimizers such that, the value of $\Phi$ along the curve $\gamma$ is negative.

The paper is organized as follow: In Section 2, we give the proof of Theorems 1.1 and 1.2. We postpone the construction of the curve $\gamma$ to Section 3.
2. **Proof of Theorems 1.1 and 1.2**

To prove Theorems 1.1 and 1.2, we consider the following truncated problem

\[
\begin{aligned}
-\Delta_p u &= f_+(x, u), \text{ in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\tag{2.1}
\]

where

\[
f_+(x, t) = \begin{cases} 
f(x, t), & t \geq 0, \\
0, & t \leq 0.
\end{cases}
\]

The solutions of (2.1) are exactly the critical points of the \(C^1\)-functional \(\Phi_+ : W^{1,p}_0(\Omega) \rightarrow \mathbb{R}\),

\[
\Phi_+(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} F_+(x, u) \, dx,
\]

where \(F_+(x, t) = \int_0^t f_+(x, s) \, ds\).

**Lemma 2.1.** Assume \((f_*)\) and \((f^1_0)\) hold, then the zero function \(0\) is not an isolated local minimizer of \(\Phi_+\).

**Proof.** For \(t \in (0, \rho)\),

\[
F_+(x, t) = F(x, t) \geq \frac{\lambda_1}{p} |t|^p.
\]

Hence for \(0 < \tau < \frac{\rho}{\max_{\Omega} \Phi_1}\),

\[
\Phi_+(\tau \Phi_1) = \frac{1}{p} \int_{\Omega} |\nabla (\tau \Phi_1)|^p \, dx - \int_{\Omega} F_+(x, \tau \Phi_1) \, dx
\]

\[
\leq \frac{\tau^p}{p} \int_{\Omega} |\nabla \Phi_1|^p \, dx - \frac{\tau^p}{p} \lambda_1 \int_{\Omega} |\Phi_1|^p \, dx = 0
\]

\[
= \Phi_+(0).
\]

Therefore, if \(0\) is a local minimizer of \(\Phi_+\), it can not be isolated. \(\Box\)

**Proof of Theorem 1.1.** Assume that \(u\) is a critical point of \(\Phi_+\). By multiplying the negative part of \(u\) to the equation in (2.1) and then integrating by parts over \(\Omega\), it is easy to see that \(u \geq 0\). Hence \(u\) is also a critical point of \(\Phi_+\).

It was shown in [13, Lemma 3.2] that under our condition \((f_\infty)\), the functional \(\Phi\) is coercive. Obviously, \(f_+\) also satisfies the condition \((f_\infty)\). Therefore, the functional \(\Phi_+\) is also coercive, thus satisfies the Palais-Smale \((PS)\) condition. It follows that \(\Phi_+\) has a minimizer \(u_+ \in W^{1,p}_0(\Omega)\). By Lemma 2.1, the zero function \(0\) is not an isolated minimizer of \(\Phi_+\). So, if \(u_+\) is an isolated critical point of \(\Phi_+\), we have \(u_+ \neq 0\). Therefore, \(u_+\) is a nonzero critical point of \(\Phi\). If \(u_+\) is not an isolated critical point of \(\Phi_+\), then \(\Phi\) already has infinitely many critical points.

Similarly, by considering \(\Phi_-\), we obtain a nonpositive critical point \(u_- \neq 0\) of \(\Phi\). Then, \(u_+\) and \(u_-\) are two nonzero solutions of (1.1). \(\Box\)

Now we are going to prove Theorem 1.2. We have obtained local minimizers \(u_{\pm}\) of \(\Phi_{\pm}\). To apply the mountain pass lemma, we will show that \(u_{\pm}\) are also local minimizers of \(\Phi\). To this purpose, let us recall the following result of García Azorero, Peral Alonso and Manfredi.
**Proposition 2.2** ([8, Theorem 1.2]). Let $f$ satisfies $(f_*)$, $\Phi$ be the functional defined by (1.2). Let $u \in C^1_0 (\Omega)$. If there exists $r > 0$ such that

$$\Phi(u) \leq \Phi(v), \quad \text{for } v \in C^1_0 (\Omega) \text{ with } \|v - u\|_{C^1_0 (\Omega)} \leq r,$$

then for some $\delta > 0$, we have

$$\Phi(u) \leq \Phi(v), \quad \text{for } v \in W^{1,p}_0 (\Omega) \text{ with } \|v - u\|_{W^{1,p}_0 (\Omega)} \leq \delta.$$

Note that a $C^1$-neighbourhood is smaller than a $W^{1,p}_0$-neighbourhood, this result is not trivial. When $p = 2$, this result is due to Brezis-Nirenberg [4].

**Proof of Theorem 1.1.** We have obtained a local minimizer $u_+$ for the functional $\Phi_+$. We know that $u_+ \geq 0$. By [2, Lemma 3.1], for any $q \geq 1$, we have $u_+ \in L^q (\Omega)$. Hence, by $(f_+)$, we also have $f_+ (\cdot, u_+ (\cdot)) \in L^q (\Omega)$ for any $q \geq 1$. In particular, we have

$$\Delta_p u_+ = -f_+ (\cdot, u_+) \in L^2 (\Omega) \subset L^2_{\text{loc}} (\Omega).$$

Since $f_+ (\cdot, u_+ (\cdot)) \in L^q (\Omega)$ for any $q > Np/(p - 1)$, the regularity results of [7, Corollary] yields $u_+ \in C^{1, \alpha}_{\text{loc}} (\Omega) \subset C^1 (\Omega)$, for some $\alpha \in (0, 1)$. Note that since $f(x, t)t \geq 0$, we also have

$$\Delta_p u_+ = -f_+ (x, u_+) \leq 0 =: \beta (u).$$

Thus by the strong maximum principle of the $p$-Laplacian [15, Theorem 5], because $u_+ \neq 0$, we deduce

$$u_+ > 0, \quad \text{in } \Omega, \quad \frac{\partial u_+}{\partial \nu} > 0, \quad \text{on } \partial \Omega, \quad (2.2)$$

where $\nu$ is the interior normal on $\partial \Omega$. Hence $u_+$ is a positive solution of (1.1).

Since $u_+$ is a local minimizer of $\Phi_+$ in $W^{1,p}_0 (\Omega)$, $u_+$ is also a local minimizer of $\Phi_+$ in $C^1_0 (\Omega)$, we can find some $r > 0$ such that

$$\Phi_+ (u_+) \leq \Phi_+(v), \quad \text{for } v \in C^1_0 (\Omega) \text{ with } \|v - u_+\|_{C^1 (\Omega)} \leq r. \quad (2.3)$$

By (2.2) we can infer that $u_+$ is an interior point of the set

$$\mathcal{P} = \{w \in C^1_0 (\Omega) : w(x) > 0 \text{ for } x \in \Omega \},$$

with respect to the $C^1$-topology. Thus we may choose $r_1 < r$ such that for $v \in C^1_0 (\Omega)$ with $\|v - u_+\|_{C^1 (\Omega)} \leq r_1$, we have $w(x) > 0$ for all $x \in \Omega$.

However, if $u > 0$, then $F(x, u(x)) = F_+(x, u(x))$. Hence $\Phi(u) = \Phi_+(u)$. Thus for $v \in C^1_0 (\Omega)$ with $\|v - u_+\|_{C^1 (\Omega)} \leq r_1$, we have

$$\Phi(u_+) = \Phi_+(u_+) \leq \Phi_+(v) = \Phi(v).$$

That is to say, $u_+$ is a local minimizer of $\Phi|_{C^1_0 (\Omega)}$, in the $C^1$-topology. Applying Proposition 2.2, we find some $\rho > 0$, such that for $v \in W^{1,p}_0 (\Omega)$ with $\|v - u_+\|_{W^{1,p}_0 (\Omega)} \leq r_1$, there holds $\Phi(u_+) \leq \Phi(v)$. Hence $u_+$ is a local minimizer of $\Phi$ in $W^{1,p}_0 (\Omega)$. Similarly, $u_-$ is a local minimizer of $\Phi$ in $W^{1,p}_0 (\Omega)$.

We may assume that $u_\pm$ are the only nonzero critical points of $\Phi_\pm$, otherwise $\Phi$ already has a third nonzero critical point. Set

$$\mathcal{A} = \{\gamma : [0, 1] \rightarrow W^{1,p}_0 (\Omega) : \gamma \text{ is continuous, } \gamma(0) = u_-, \gamma(1) = u_+ \}$$
and define
\[ c = \inf_{\gamma \in \mathcal{A}} \sup_{t \in [0,1]} \Phi(\gamma(t)). \tag{2.4} \]
Since \( \Phi \) satisfies the \( (PS) \) condition, \( u_{\pm} \) are local minimizers of \( \Phi \), by the mountain pass lemma we see that \( c \) is a critical value of \( \Phi \),
\[ c > \Phi(u_+), \quad c > \Phi(u_-). \]
Thus \( \Phi \) has a critical point \( u \neq u_{\pm} \), such that \( \Phi(u) = c \).

Note that \( \Phi(0) = 0 \), in the next section, we will construct a curve \( \gamma \in \mathcal{A} \) such that the value of \( \Phi \) along \( \gamma \) is negative. Hence \( c < 0 \) and the third critical point \( u \) is nonzero. Therefore, the functional \( \Phi \) has three nonzero critical points \( u_+ \), \( u_- \) and \( u \). This completes the proof of Theorem 1.2.

\[ \square \]

3. The construction of the curve \( \gamma \)

In this section, we will construct the curve \( \gamma \) mentioned in the proof of Theorem 1.2. Let
\[ I(u) = \int_{\Omega} |\nabla u|^p \, dx, \quad \mathcal{A} = \left\{ u \in W^{1,p}_0(\Omega) : \int_{\Omega} |u|^p \, dx = 1 \right\}. \]
For \( b \in \mathbb{R} \), we denote
\[ \mathcal{O}_b = \{ u \in \mathcal{A} : I(u) < b \}. \]
As a special case of Dancer-Perera [6, Lemma 2.6], we have

**Proposition 3.1.** The set \( \mathcal{O}_b \) is path-connected if and only if \( b > \lambda_2 \).

**Remark 3.2.** In [6], this topological property of \( \mathcal{O}_b \) and the limits (1.3) are used to show that the critical group \( C_1(\Phi, 0) = 0 \). But the critical point \( u \) obtained by the mountain pass lemma satisfies \( C_1(\Phi, u) \neq 0 \). Thus \( u \neq 0 \) is the third nonzero solution of (1.1). Here, since we do not assume (1.3), we can not deduce \( C_1(\Phi, 0) = 0 \) and distinguish \( 0 \) and \( u \).

For the definition of critical group, the reader is referred to Chang [5].

Obviously, for \( \lambda > \lambda_2 \), the first eigenfunctions \( \pm \phi_1 \in \mathcal{O}_\lambda \). Since \( \mathcal{O}_\lambda \) is path-connected, we can find a continuous curve \( \ell : [0,1] \to \mathcal{O}_\lambda \) such that \( \ell(0) = -\phi_1, \ell(1) = \phi_1 \).

Now, by \( (f_\ast) \) and \( (f_0^2) \), there exists \( C > 0 \) such that for \( x \in \Omega \) and \( t \in \mathbb{R} \), we have
\[ F(x, t) \geq \frac{\lambda}{p} |t|^p - C |t|^\theta. \tag{3.1} \]
Let
\[ A := \sup_{u \in \ell([0,1])} \left\{ \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{p} \right\}, \quad B = \sup_{u \in \ell([0,1])} \int_{\Omega} |u|^\theta \, dx. \]
Since \([0,1]\) is compact, we see that \( A, B \) are finite. Moreover, it is easy to see that \( A < 0 \). Noting that \( p < \theta \), we can choose \( s > 0 \) small enough, such that
\[ As^p + CBs^\theta < 0. \tag{3.2} \]
Now, for \( u \in \ell([0,1]) \), by (3.1) and (3.2) we obtain
\[ \Phi(su) = \frac{1}{p} \int_{\Omega} |\nabla (su)|^p \, dx - \int_{\Omega} F(x, su) \, dx \]
Thus, we obtain a curve \( \gamma_0 : [0, 1] \rightarrow W_0^{1,p}(\Omega) \) defined by \( \gamma_0(t) = s\ell(t) \) such that

\[
\gamma_0(0) = -s\phi_1, \quad \gamma_0(1) = s\phi_1, \quad \Phi(\gamma_0(t)) < 0 \quad \text{for} \ t \in [0, 1].
\]

Next, we will connect \( \pm s\phi_1 \) to \( u_\pm \) by continuous curves \( \gamma_\pm : [0, 1] \rightarrow W_0^{1,p}(\Omega) \), such that \( \Phi(\gamma_\pm(t)) < 0 \). These curves are constructed by the following ’second deformation lemma’.

**Proposition 3.3** ([5, 16]). _Let \( X \) be a Banach space, \( \varphi \in C^1(X, \mathbb{R}) \) satisfies the (PS) condition and has only finitely many critical points, \( a \) is the unique critical value of \( \varphi \) in \( [a, b] \subset \mathbb{R} \). Then there exists a continuous map \( \eta : [0, 1] \times (\varphi_b \setminus \mathcal{K}_b) \rightarrow \varphi_b \setminus \mathcal{K}_b \), such that

(i) \( \eta(0, u) = u \), for any \( u \in \varphi_b \setminus \mathcal{K}_b \),

(ii) \( \eta(t, u) = u \), for \( t, u \in [0, 1] \times \varphi^a \),

(iii) \( \eta(1, \varphi_b \setminus \mathcal{K}_b) \subset \varphi^a \),

(iv) for any \( u \in \varphi_b \setminus \mathcal{K}_b \), \( \varphi(\eta(t, u)) \) is decreasing in \( t \in [0, 1] \).

where \( \varphi^a := \varphi^{-1}(-\infty, a) \), \( \mathcal{K}_b := \{ u \in \varphi^{-1}(b) : \varphi'(u) = 0 \} \).

Since we have assumed that the only nonzero critical point of \( \Phi_+ \) is \( u_+ \), let \( a = \Phi_+(u_+) \), then \( a = \inf_{W_0^{1,p}(\Omega)} \Phi_+ \). Thus \( \Phi_+^a \subset \{ u_+ \} \). Moreover, by (3.3) we see that

\[
\Phi_+(s\phi_1) = \Phi(s\phi_1) < 0,
\]

that is, \( s\phi_1 \in \Phi_+^0 \setminus \mathcal{K}_b \). By the above inequality we also know that \( a < 0 \) and \( \Phi_+^a = \{ u_+ \} \).

Now applying Proposition 3.3 to \( \Phi_+ \) we obtain a continuous map

\[
\eta : [0, 1] \times (\Phi^+_0 \setminus \mathcal{K}_0) \rightarrow \Phi^+_0 \setminus \mathcal{K}_0
\]

such that the properties (i)–(iv) in Proposition 3.3 hold for \( \Phi_+ \). Let \( \gamma_+(t) = \eta(t, s\phi_1) \), we obtain a curve \( \gamma_+ : [0, 1] \rightarrow W_0^{1,p}(\Omega) \) such that \( \gamma_+(0) = s\phi_1, \gamma_+(1) = u_+ \) and for \( t \in [0, 1] \)

\[
\Phi_+(\gamma_+(t)) = \Phi_+(\eta(t, s\phi_1)) \leq \Phi_+(\eta(0, s\phi_1)) = \Phi_+(s\phi_1) < 0.
\] (3.4)

By our assumption, \( f(x, u) \geq 0 \) for \( x \in \Omega \) and \( t \in \mathbb{R} \), hence \( F(x, u) \geq 0 \) and we have

\[
\Phi(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{1}{p} \int_{\Omega} |u|^p \, dx + C s^\theta \int_{\Omega} |u|^\theta \, dx
\]

\[
= \left( \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{\lambda}{p} \right) s^p + C \left( \int_{\Omega} |u|^\theta \, dx \right)^\theta
\]

\[
\leq As^p + CBS^\theta < 0. \quad (3.3)
\]

Thus, by (3.4) we see that \( \Phi(\gamma_+(t)) < 0 \) for \( t \in [0, 1] \).

Similarly, we construct a curve \( \gamma_- \) between \(-s\phi_1 \) and \( u_- \). Jointing the curves \( \gamma_- \), \( \gamma_0 \) and \( \gamma_+ \), we obtain a curve \( \gamma \in \mathcal{A} \) such that \( \sup_{t \in [0, 1]} \Phi(\gamma(t)) < 0 \). Hence the critical value \( c \) given by (2.4) is negative, and the corresponding critical point \( u \) is nonzero.
References


