Multiple Solutions for Coercive p-Laplacian Equations

Shibo Liu

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China

Abstract

We obtain multiple nonzero solutions for coercive p-Laplacian equations. In order to obtain the third nonzero solution, we use the second deformation lemma to construct the desired mountain pass path.

Key words: p-Laplacian, minimizers, mountain pass lemma, the second deformation lemmas

1. Introduction

In this paper, we consider the existence of multiple nonzero solutions of the Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $p > 1$, $-\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\,\nabla u)$ denotes the p-Laplacian operator, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

We assume that the nonlinearity $f(x,u)$ satisfies the following condition

(f_*) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth

$$|f(x,t)| \le C_1(1 + |t|^{\theta-1}), \qquad \text{for a.e. } x \in \Omega,\ t \in \mathbb{R},$$

where $C_1 > 0$, $\theta \in (p, p^*)$ and $p^* = Np/(N-p)$ for $p < N$, while $p^* = +\infty$ if $p \ge N$.

Then, it is well known that the weak solutions of the problem (1.1) are exactly the critical points of the C^1-functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x, \tag{1.2}$$

where $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$, and $W_0^{1,p}(\Omega)$ is the well-known Sobolev space endowed with the norm $\|u\| = \left(\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)^{1/p}$. For $q \ge 1$, we denote by $|u|_q = \left(\int_\Omega |u|^q\,\mathrm{d}x\right)^{1/q}$ the usual Lebesgue norm on $L^q(\Omega)$.

Email address: liusb@math.pku.edu.cn (Shibo Liu).

Let λ_1 and λ_2 be the first and the second eigenvalues of $-\Delta_p$ on $W_0^{1,p}(\Omega)$. It is well known that $\lambda_1 > 0$ is a simple eigenvalue with fixed-sign eigenfunctions. By ϕ_1 we denote the positive eigenfunction of λ_1 such that $|\phi_1|_p = 1$. We also know that $\sigma(-\Delta_p) \cap (\lambda_1, \lambda_2) = \emptyset$, where $\sigma(-\Delta_p)$ is the spectrum of $-\Delta_p$, see [9].

In this paper, we will study the existence of multiple solutions of the problem (1.1) in the case that Φ is coercive. More precisely, we have the following result.

Theorem 1.1. *Assume that* (f_*) *holds. If in addition we have*

(f_0^1) *there exists* $\rho > 0$ *such that* $F(x,t) \geq \dfrac{\lambda_1}{p}|t|^p$ *for* $|t| \leq \rho$ *and* $x \in \Omega$*,*

(f_∞) $\displaystyle\limsup_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} < \lambda_1$,

then the problem (1.1) has at least two nonzero solutions.

Recall that assuming (f_*), (f_∞) and

(f_0^*) there exist $\rho > 0$ and $\bar{\lambda} \in (\lambda_1, \lambda_2)$ such that for $|t| \leq \rho$ and $x \in \Omega$ there holds

$$\lambda_1 |t|^p \leq pF(x,t) \leq \bar{\lambda}\, |t|^p ,$$

the existence of two nonzero solutions of the problem (1.1) has been obtained by Liu-Su [13, Theorem 1.1]. Here, our condition (f_0^1) is much weaker than their (f_0^*). In their argument, the condition (f_0^*) is used to produce a *local linking* at zero, then the famous three critical points theorem [3, 11] is applied. Our proof of Theorem 1.1 is completely different. We will use the truncated method to produce a nonpositive solution and a nonnegative one for our problem (1.1).

If we impose stronger conditions on the nonlinearity $f(x,u)$, we can obtain one more solution.

Theorem 1.2. *Assume that* (f_*) *and* (f_∞) *hold,* $f(x,t)t \geq 0$ *for* $x \in \Omega$ *and* $t \in \mathbb{R}$*. If in addition we have*

(f_0^2) *there exist* $\rho > 0$ *and* $\lambda > \lambda_2$ *such that* $F(x,t) \geq \dfrac{\lambda}{p}|t|^p$ *for* $|t| \leq \rho$

then the problem (1.1) has at least three nonzero solutions.

We refer to [6, 10, 17] for recent work on the existence of three nonzero solutions for coercive elliptic equations. In all these papers the asymptotic limits

$$a_\pm = \lim_{t\to 0\pm} \frac{f(x,t)}{|t|^{p-2}t} \tag{1.3}$$

play an essential role. Here we do not require that the limits exist. For the case that f is 'superlinear' at zero, that is, $F(x,t) \geq a|t|^\mu$ for small $|t|$ and some $\mu \in (1,p)$, see [14] for $p = 2$ and [12] for general $p > 1$. Obviously, this 'superlinear' condition is stronger than (f_0^2).

Now let us say a few words about the proof of Theorem 1.2. It turns out that the two nonzero solutions obtained in Theorem 1.1 are local minimizers of the functional Φ. Naturally, the third solution will be found via the mountain pass lemma [1]. However, we have to show that this solution is not the trivial one, i.e., the zero function $\mathbf{0}$. To this end, we will construct a curve γ between the two local minimizers such that, the value of Φ along the curve γ is negative.

The paper is organized as follow: In Section 2, we give the proof of Theorems 1.1 and 1.2. We postpone the construction of the curve γ to Section 3.

2. Proof of Theorems 1.1 and 1.2

To prove Theorems 1.1 and 1.2, we consider the following truncated problem

$$\begin{cases} -\Delta_p u = f_+(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where

$$f_+(x,t) = \begin{cases} f(x,t), & t \geq 0, \\ 0, & t \leq 0. \end{cases}$$

The solutions of (2.1) are exactly the critical points of the C^1-functional $\Phi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$\Phi_+(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \int_\Omega F_+(x,u)\,\mathrm{d}x,$$

where $F_+(x,t) = \int_0^t f_+(x,s)\,\mathrm{d}s$.

Lemma 2.1. *Assume* (f_*) *and* (f_0^1) *hold, then the zero function* $\mathbf{0}$ *is not an isolated local minimizer of* Φ_+.

Proof. For $t \in (0,\rho)$,

$$F_+(x,t) = F(x,t) \geq \frac{\lambda_1}{p}|t|^p.$$

Hence for $0 < \tau < \dfrac{\rho}{\max_{\overline{\Omega}} \phi_1}$,

$$\begin{aligned} \Phi_+(\tau\phi_1) &= \frac{1}{p}\int_\Omega |\nabla(\tau\phi_1)|^p \,\mathrm{d}x - \int_\Omega F_+(x,\tau\phi_1)\,\mathrm{d}x \\ &\leq \frac{\tau^p}{p}\int_\Omega |\nabla\phi_1|^p \,\mathrm{d}x - \frac{\tau^p}{p}\lambda_1\int_\Omega |\phi_1|^p \,\mathrm{d}x = 0 \\ &= \Phi_+(\mathbf{0}). \end{aligned}$$

Therefore, if $\mathbf{0}$ is a local minimizer of Φ_+, it can not be isolated. □

Proof of Theorem 1.1. Assume that u is a critical point of Φ_+. By multiplying the negative part of u to the equation in (2.1) and then integrating by parts over Ω, it is easy to see that $u \geq 0$. Hence u is also a critical point of Φ.

It was shown in [13, Lemma 3.2] that under our condition (f_∞), the functional Φ is coercive. Obviously, f_+ also satisfies the condition (f_∞). Therefore, the functional Φ_+ is also coercive, thus satisfies the Palais-Smale (PS) condition. It follows that Φ_+ has a minimizer $u_+ \in W_0^{1,p}(\Omega)$. By Lemma 2.1, the zero function $\mathbf{0}$ is not an isolated minimizer of Φ_+. So, if u_+ is an isolated critical point of Φ_+, we have $u_+ \neq \mathbf{0}$. Therefore, u_+ is a nonzero critical point of Φ. If u_+ is not an isolated critical point of Φ_+, then Φ already has infinitely many critical points.

Similarly, by considering Φ_-, we obtain a nonpositive critical point $u_- \neq \mathbf{0}$ of Φ. Then, u_+ and u_- are two nonzero solutions of (1.1). □

Now we are going to prove Theorem 1.2. We have obtained local minimizers $u_\pm$ of $\Phi_\pm$. To apply the mountain pass lemma, we will show that $u_\pm$ are also local minimizers of Φ. To this purpose, let us recall the following result of García Azorero, Peral Alonso and Manfredi.

Proposition 2.2 ([8, Theorem 1.2]). *Let f satisfies (f_*), Φ be the functional defined by (1.2). Let $u \in C_0^1(\Omega)$. If there exists $r > 0$ such that*

$$\Phi(u) \le \Phi(v), \qquad \text{for } v \in C_0^1(\Omega) \text{ with } \|v-u\|_{C_0^1(\Omega)} \le r,$$

then for some $\delta > 0$, we have

$$\Phi(u) \le \Phi(v), \qquad \text{for } v \in W_0^{1,p}(\Omega) \text{ with } \|v-u\|_{W_0^{1,p}(\Omega)} \le \delta.$$

Note that a C^1-neighbourhood is smaller than a $W_0^{1,p}$-neighbourhood, this result is not trivial. When $p = 2$, this result is due to Brezis-Nirenberg [4].

Proof of Theorem 1.2. We have obtained a local minimizer u_+ for the functional Φ_+. We know that $u_+ \ge 0$. By [2, Lemma 3.1], for any $q \ge 1$, we have $u_+ \in L^q(\Omega)$. Hence, by (f_*), we also have $f_+(\cdot, u_+(\cdot)) \in L^q(\Omega)$ for any $q \ge 1$. In particular, we have

$$\Delta_p u_+ = -f_+(\cdot, u_+) \in L^2(\Omega) \subset L^2_{\mathrm{loc}}(\Omega).$$

Since $f_+(\cdot, u_+(\cdot)) \in L^q(\Omega)$ for any $q > Np/(p-1)$, the regularity results of [7, Corollary] yields $u_+ \in C^{1,\alpha}_{\mathrm{loc}}(\Omega) \subset C^1(\Omega)$, for some $\alpha \in (0,1)$. Note that since $f(x,t)t \ge 0$, we also have

$$\Delta_p u_+ = -f_+(x, u_+) \le 0 =: \beta(u).$$

Thus by the strong maximum principle of the p-Laplacian [15, Theorem 5], because $u_+ \not\equiv 0$, we deduce

$$u_+ > 0, \quad \text{in } \Omega, \qquad \frac{\partial u_+}{\partial \nu} > 0, \quad \text{on } \partial\Omega, \tag{2.2}$$

where ν is the interior normal on $\partial\Omega$. Hence u_+ is a positive solution of (1.1).

Since u_+ is a local minimizer of Φ_+ in $W_0^{1,p}(\Omega)$, u_+ is also a local minimizer of Φ_+ in $C_0^1(\Omega)$, we can find some $r > 0$ such that

$$\Phi_+(u_+) \le \Phi_+(v), \qquad \text{for } v \in C_0^1(\Omega) \text{ with } \|v-u_+\|_{C^1(\Omega)} \le r. \tag{2.3}$$

By (2.2) we can infer that u_+ is an interior point of the set

$$\mathscr{P} = \left\{ w \in C_0^1(\Omega) : \ w(x) > 0 \text{ for } x \in \Omega \right\},$$

with respect to the C^1-topology. Thus we may choose $r_1 < r$ such that for $v \in C_0^1(\Omega)$ with $\|v-u_+\|_{C^1(\Omega)} \le r_1$, we have $v(x) > 0$ for all $x \in \Omega$.

However, if $u > 0$, then $F(x,u(x)) = F_+(x,u(x))$. Hence $\Phi(u) = \Phi_+(u)$. Thus for $v \in C_0^1(\Omega)$ with $\|v-u_+\|_{C^1(\Omega)} \le r_1$, we have

$$\Phi(u_+) = \Phi_+(u_+) \le \Phi_+(v) = \Phi(v).$$

That is to say, u_+ is a local minimizer of $\Phi|_{C_0^1(\Omega)}$, in the C^1-topology. Applying Proposition 2.2, we find some $\rho > 0$, such that for $v \in W_0^{1,p}(\Omega)$ with $\|v-u_+\|_{W_0^{1,p}(\Omega)} \le r_1$, there holds $\Phi(u_+) \le \Phi(v)$. Hence u_+ is a local minimizer of Φ in $W_0^{1,p}(\Omega)$. Similarly, u_- is a local minimizer of Φ in $W_0^{1,p}(\Omega)$.

We may assume that $u_\pm$ are the only nonzero critical points of $\Phi_\pm$, otherwise Φ already has a third nonzero critical point. Set

$$\mathscr{A} = \left\{ \gamma : [0,1] \to W_0^{1,p}(\Omega) : \ \gamma \text{ is continuous}, \gamma(0) = u_-, \gamma(1) = u_+ \right\}$$

and define

$$c = \inf_{\gamma \in \mathscr{A}} \sup_{t \in [0,1]} \Phi(\gamma(t)). \tag{2.4}$$

Since Φ satisfies the (PS) condition, $u_\pm$ are local minimizers of Φ, by the mountain pass lemma we see that c is a critical value of Φ,

$$c > \Phi(u_+), \qquad c > \Phi(u_-).$$

Thus Φ has a critical point $u \neq u_\pm$, such that $\Phi(u) = c$.

Note that $\Phi(\mathbf{0}) = 0$, in the next section, we will construct a curve $\gamma \in \mathscr{A}$ such that the value of Φ along γ is negative. Hence $c < 0$ and the third critical point u is nonzero. Therefore, the functional Φ has three nonzero critical points u_+, u_- and u. This completes the proof of Theorem 1.2. □

3. The construction of the curve γ

In this section, we will construct the curve γ mentioned in the proof of Theorem 1.2. Let

$$I(u) = \int_\Omega |\nabla u|^p \, dx, \qquad \mathscr{S} = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^p \, dx = 1 \right\}.$$

For $b \in \mathbb{R}$, we denote

$$\mathscr{O}_b = \{u \in \mathscr{S} : I(u) < b\}.$$

As a special case of Dancer-Perera [6, Lemma 2.6], we have

Proposition 3.1. *The set $\mathscr{O}_b$ is path-connected if and only if $b > \lambda_2$.*

Remark 3.2. In [6], this topological property of $\mathscr{O}_b$ and the limits (1.3) are used to show that the critical group $C_1(\Phi, \mathbf{0}) = 0$. But the critical point u obtained by the mountain pass lemma satisfies $C_1(\Phi, u) \neq 0$. Thus $u \neq \mathbf{0}$ is the third nonzero solution of (1.1). Here, since we do not assume (1.3), we can not deduce $C_1(\Phi, \mathbf{0}) = 0$ and distinguish $\mathbf{0}$ and u.

For the definition of critical group, the reader is referred to Chang [5].

Obviously, for $\lambda > \lambda_2$, the first eigenfunctions $\pm\phi_1 \in \mathscr{O}_\lambda$. Since $\mathscr{O}_\lambda$ is path-connected, we can find a continuous curve $\ell : [0,1] \to \mathscr{O}_\lambda$ such that $\ell(0) = -\phi_1$, $\ell(1) = \phi_1$.

Now, by (f_*) and (f_0^2), there exists $C > 0$ such that for $x \in \Omega$ and $t \in \mathbb{R}$, we have

$$F(x,t) \geq \frac{\lambda}{p} |t|^p - C\,|t|^\theta. \tag{3.1}$$

Let

$$A := \sup_{u \in \ell([0,1])} \left\{ \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\lambda}{p} \right\}, \qquad B = \sup_{u \in \ell([0,1])} \int_\Omega |u|^\theta \, dx.$$

Since $[0,1]$ is compact, we see that A, B are finite. Moreover, it is easy to see that $A < 0$. Noting that $p < \theta$, we can choose $s > 0$ small enough, such that

$$As^p + CBs^\theta < 0. \tag{3.2}$$

Now, for $u \in \ell([0,1])$, by (3.1) and (3.2) we obtain

$$\Phi(su) = \frac{1}{p} \int_\Omega |\nabla(su)|^p \, dx - \int_\Omega F(x, su) \, dx$$

$$\begin{aligned}
&\le \frac{s^p}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{s^p}{p}\lambda\int_\Omega |u|^p \,\mathrm{d}x + Cs^\theta\int_\Omega |u|^\theta \,\mathrm{d}x \\
&= \left(\frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{p}\right)s^p + C\left(\int_\Omega |u|^\theta \,\mathrm{d}x\right)s^\theta \\
&\le As^p + CBs^\theta < 0. \qquad (3.3)
\end{aligned}$$

Thus, we obtain a curve $\gamma_0 : [0,1] \to W_0^{1,p}(\Omega)$ defined by $\gamma_0(t) = s\ell(t)$ such that

$$\gamma_0(0) = -s\phi_1, \quad \gamma_0(1) = s\phi_1, \quad \Phi(\gamma_0(t)) < 0 \quad \text{for } t \in [0,1].$$

Next, we will connect $\pm s\phi_1$ to $u_\pm$ by continuous curves $\gamma_\pm : [0,1] \to W_0^{1,p}(\Omega)$, such that $\Phi(\gamma_\pm(t)) < 0$. These curves are constructed by the following 'second deformation lemma'.

Proposition 3.3 ([5, 16])**.** *Let X be a Banach space, $\varphi \in C^1(X,\mathbb{R})$ satisfies the (PS) condition and has only finitely many critical points, a is the unique critical value of φ in $[a,b) \subset \mathbb{R}$. Then there exists a continuous map $\eta : [0,1] \times (\varphi^b \backslash \mathscr{K}_b) \to \varphi^b \backslash \mathscr{K}_b$, such that*

(i) $\eta(0,u) = u$, *for any* $u \in \varphi^b \backslash \mathscr{K}_b$,
(ii) $\eta(t,u) = u$, *for* $(t,u) \in [0,1] \times \varphi^a$,
(iii) $\eta(1,\varphi^b \backslash \mathscr{K}_b) \subset \varphi^a$,
(iv) *for any* $u \in \varphi^b \backslash \mathscr{K}_b$, $\varphi(\eta(t,u))$ *is decreasing in* $t \in [0,1]$.

where $\varphi^a := \varphi^{-1}(-\infty,a]$, $\mathscr{K}_b := \{u \in \varphi^{-1}(b) : \varphi'(u) = 0\}$.

Since we have assumed that the only nonzero critical point of Φ_+ is u_+, let $a = \Phi_+(u_+)$, then $a = \inf_{W_0^{1,p}(\Omega)} \Phi_+$. Thus $\Phi_+^a \subset \{u_+, \mathbf{0}\}$. Moreover, by (3.3) we see that

$$\Phi_+(s\phi_1) = \Phi(s\phi_1) < 0,$$

that is, $s\phi_1 \in \Phi_+^0 \backslash \mathscr{K}_0$. By the above inequality we also know that $a < 0$ and $\Phi_+^a = \{u_+\}$.

Now applying Proposition 3.3 to Φ_+ we obtain a continuous map

$$\eta : [0,1] \times (\Phi_+^0 \backslash \mathscr{K}_0) \to \Phi_+^0 \backslash \mathscr{K}_0$$

such that the properties (i)–(iv) in Proposition 3.3 hold for Φ_+. Let $\gamma_+(t) = \eta(t, s\phi_1)$, we obtain a curve $\gamma_+ : [0,1] \to W_0^{1,p}(\Omega)$ such that $\gamma_+(0) = s\phi_1$, $\gamma_+(1) = u_+$ and for $t \in [0,1]$

$$\Phi_+(\gamma_+(t)) = \Phi_+(\eta(t,s\phi_1)) \le \Phi_+(\eta(0,s\phi_1)) = \Phi_+(s\phi_1) < 0. \qquad (3.4)$$

By our assumption, $f(x,t)t \ge 0$ for $x \in \Omega$ and $t \in \mathbb{R}$, hence $F(x,t) \ge 0$ and we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x \\
&= \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \int_{u\ge 0} F(x,u)\,\mathrm{d}x - \int_{u\le 0} F(x,u)\,\mathrm{d}x \\
&\le \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \int_{u\ge 0} F(x,u)\,\mathrm{d}x \\
&= \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \int_\Omega F_+(x,u)\,\mathrm{d}x = \Phi_+(u).
\end{aligned}$$

Thus, by (3.4) we see that $\Phi(\gamma_+(t)) < 0$ for $t \in [0,1]$.

Similarly, we construct a curve γ_- between $-s\phi_1$ and u_-. Jointing the curves γ_-, γ_0 and γ_+, we obtain a curve $\gamma \in \mathscr{A}$ such that $\sup_{t\in[0,1]} \Phi(\gamma(t)) < 0$. Hence the critical value c given by (2.4) is negative, and the corresponding critical point u is nonzero.

References

[1] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis*, 14 (1973), 349–381.

[2] T. Bartsch, Z. L. Liu, On a superlinear elliptic p-Laplacian equation, *J. Differential Equations*, 198 (2004) 149–175.

[3] H. Brezis, L. Nirenberg, Remarks on finding critical points, *Comm. Pure Appl. Math.*, 44 (1991), 939–963.

[4] H. Brezis, L. Nirenberg, H^1 versus C^1 local minimizers, *C. R. Acad. Sci. Paris Ser. I Math.*, 317 (1993), 465–472.

[5] K. C. Chang, Infinite dimensional Morse theory and multiple solution problem, Birkhäuser, Bostom, 1993.

[6] N. Dancer, K. Perera, Some remarks on the Fučík spectrum of the p-Laplacian and critical groups. *J. Math. Anal. Appl.* 254 (2001), no. 1, 164–177.

[7] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.*, 7 (1983), 827–850.

[8] J. P. García Azorero, I. Peral Alonso, J. Manfredi, Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations, *Commun. Contemp. Math.*, 2 (2000), 385–404.

[9] P. Lindqvist, On the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2} u = 0$, *Proc. Amer. Math. Soc.*, 109 (1990), 609–623.

[10] S. J. Li, Z. T. Zhang, Sign-changing and multiple solutions theorems for semilinear elliptic boundary value problems with jumping nonlinearities, *Acta Math. Sin.*, 16 (2001), 113–122.

[11] J. Q. Liu, S. J. Li, An existence theorem for multiple critical points and its application. (Chinese) *Kexue Tongbao* (Chinese), 29 (1984), 1025–1027.

[12] J. Q. Liu, S. B. Liu, The existence of multiple solutions for quasilinear elliptic equations, *Bull. London Math. Soc.*, (to appear).

[13] J. Q. Liu, J. B. Su, Remarks on multiple nontrivial solutions for quasilinear resonant problems, *J. Math. Anal. Appl.*, 258 (2001), 209–222.

[14] V. Moroz, Solutions of superlinear at zero elliptic equations via Morse theory, *Topol. Methods. Nonlinear Anal.*, 10 (1997) 387C398.

[15] J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.*, 12 (1984), no. 3, 191–202.

[16] Z. Q. Wang, A note on the second deformation theorem. (Chinese) *Acta Math. Sinica*, 30 (1987), 106–C110.

[17] Z. T. Zhang, J. Q. Chen, S. J. Li, Construction of pseudo-gradient vector field and sign-changing multiple solutions involving p-Laplacian, *J. Differential Equations*, 201 (2004), 287–303.