

## Multiple periodic solutions for non-linear difference systems involving the $p$ -Laplacian

Shibo Liu\*

*Department of Mathematics, Shantou University, Shantou 515063, PR China*

*(Received 10 January 2009; final version received 15 February 2010)*

Using the three critical points theorem, Clark's theorem and the Morse theory, multiple periodic solutions for non-linear difference systems involving the  $p$ -Laplacian are obtained by variational methods.

**Keywords:** non-linear difference systems; local linking; three critical points theorem; critical groups; Clark's theorem

**AMS Subject Classification:** 39A10; 39A11; 58E05

### 1. Introduction

Let  $m > 1$  be a fixed integer, we consider the existence of multiple  $m$ -periodic solutions for non-linear difference systems of the form

$$\Delta(|\Delta x_{n-1}|^{p-2} \Delta x_{n-1}) + f(n, x_{n+1}, x_n, x_{n-1}) = 0, \quad x_{n+m} = x_n, \quad n \in \mathbf{Z}; \quad (1.1)$$

where  $p > 1$ ,  $\Delta$  is the forward difference operator defined by  $\Delta x_n = x_{n+1} - x_n$ ,  $f : \mathbf{Z} \times \mathbf{R}^{3N} \rightarrow \mathbf{R}^N$  is continuous and there exists  $F(\cdot, \cdot, \cdot) \in C^1(\mathbf{Z} \times \mathbf{R}^{2N}, \mathbf{R})$  such that

$$f(n, u, v, w) = F'_2(n-1, v, w) + F'_3(n, u, v);$$

**Q1**

here  $F'_i$  denotes the partial derivative of  $F$  with respect to the  $i$ th variable. When  $p = 2$  and  $F(t, u, v)$  is independent of  $u$ , that is  $F(t, u, v) = G(t, v)$ , for some  $G \in C^1(\mathbf{Z} \times \mathbf{R}^N, \mathbf{R})$ , we have  $F'_2 = \partial_u F = 0$ . Hence,

$$f(n, x_{n+1}, x_n, x_{n-1}) = F'_3(n, x_{n+1}, x_n) = \partial_v G(n, x_n) =: G'(n, x_n).$$

We see that the system in (1.1) reduces to the second-order discrete Hamiltonian system

$$\Delta^2 x_{n-1} + G'(n, x_n) = 0,$$

which has been discussed in [2,13].

---

\*Email: liusb@stu.edu.cn

ISSN 1023-6198 print/ISSN 1563-5120 online

© 2010 Taylor & Francis

DOI: 10.1080/10236191003730480

<http://www.informaworld.com>

We may consider the problem (1.1) as a discrete analogue of the following non-linear functional differential equation involving the  $p$ -Laplacian:

$$(|x'|^{p-2}x')' + f(t, x(t+1), x(t), x(t-1)) = 0, \quad t \in \mathbb{R}.$$

This kind of equation arises in the study of the existence of solitary waves of lattice differential equations, see [12] and the references therein.

To state our main results, we denote by  $c$  the real number such that

$$c^{-1} = \min_{x,y>0, x+y=1} (x^p + y^p) > 0. \tag{1.2}$$

We assume that the function  $F$  satisfies the following conditions:

( $F_*$ )  $F \in C^1(\mathbb{Z} \times \mathbb{R}^{2N}, \mathbb{R})$ ,  $F(n, 0, 0) = 0$ , and

$$F(n, u, v) = F(n+m, u, v), \quad \text{for all } (n, u, v) \in \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N;$$

( $F_0$ ) there exists some  $\eta > 0$  such that  $F(n, u, v) \geq 0$  for  $|u| + |v| \leq 2\eta$ , and

$$\lim_{|u|+|v| \rightarrow 0} \frac{F(n, u, v)}{|u|^p + |v|^p} = 0, \quad \text{uniformly in } n \in \mathbb{Z}; \tag{1.3}$$

( $F_\infty$ ) there exist  $a_1 > p^{-1}c$  and  $a_2 > 0$  such that

$$F(n, u, v) \geq a_1(|u|^p + |v|^p) - a_2, \quad \text{for all } (n, u, v) \in \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N. \tag{1.4}$$

It follows from (1.3) that  $f(n, 0, 0, 0) = 0$ ; therefore,  $x_n = 0$  is a trivial  $m$ -periodic solution of the problem (1.1). Therefore, we want to find non-trivial solutions. The main result of this paper is the following theorems.

**THEOREM 1.1.** *Suppose that ( $F_*$ ), ( $F_0$ ) and ( $F_\infty$ ) hold, then the problem (1.1) has at least two non-trivial  $m$ -periodic solutions.*

In a recent paper (Chen and Fang [5]), the scalar case  $N = 1$  has been considered by variational methods. Under ( $F_*$ ), (1.3) and some conditions slightly stronger than ( $F_\infty$ ), as well as  $F(n, u, v) \geq 0$  for all  $(n, u, v) \in \mathbb{Z} \times \mathbb{R} \times \mathbb{R}$ , they obtained two non-zero periodic solutions by the linking theorem [11] (Theorem 5.3). In that paper,  $F(n, u, v) \geq 0$  is a global requirement, which is crucial for applying the linking theorem.

The novelty of our Theorem 1.1 is that, we only require  $F(n, u, v) \geq 0$  for small  $|u|$  and  $|v|$ . In this case, the linking theorem is no longer applicable. Our proof of Theorem 1.1 is also based on variational methods. However, instead of the linking theorem, we will use the three critical points theorem of Brezis and Nirenberg [3] and Liu and Li [9]. It turns out that this approach is considerably simpler.

We may also consider the case that  $F(n, u, v) \leq 0$  for small  $|u|$  and  $|v|$ . In this situation, we have the following result, which is obtained by the Morse theory.

**THEOREM 1.2.** *Suppose that ( $F_*$ ), ( $F_\infty$ ) hold. If there exists  $\eta > 0$  such that  $F(n, u, v) \leq 0$  for  $|u| + |v| \leq 2\eta$ , then the problem (1.1) has at least two non-trivial  $m$ -periodic solutions.*

It turns out again that  $x_n = 0$  is a trivial  $m$ -periodic solution of the problem (1.1). We emphasize that, in Theorem 1.2 we do not require the limit condition (1.3). In our next

99 result, we consider the case that  $F(n, u, v)$  is even in  $(u, v)$ , then we can obtain more  
 100 solutions.

101  
 102 **THEOREM 1.3.** *Suppose that  $F(n, u, v) = F(n, -u, -v)$  for all  $(n, u, v) \in \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N$ . If  $F$   
 103 satisfies  $(F_*)$ ,  $(F_0)$  and  $(F_\infty)$ , then the problem (1.1) has at least  $(m - 1)N$  pairs of non-  
 104 trivial  $m$ -periodic solutions.  
 105*

106 This symmetric case has not been considered in [5]. The proof of this theorem is based  
 107 on variational methods and Clark’s theorem [6].

108 The variational methods and critical point theory have been extensively applied to  
 109 differential equations, see Rabinowitz [11] and Mawhin–Willem [10] for an excellent  
 110 survey. Since the appearance of [7], in recent years, this approach has also been used to  
 111 study difference equations by many authors, see for example [2,5,8,13].  
 112

113 **2. Variational framework**

114 Following [2], for our fixed integer  $m > 1$ , we define the linear operations on  
 115

116 
$$E_m = \{x = \{x_n\}_{n \in \mathbb{Z}} : x_n \in \mathbb{R}^N, \quad x_{n+m} = x_n, \quad n \in \mathbb{Z}\},$$

117  
 118 in an obvious way, and then define the inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$  on  $E_m$  as follows:  
 119

120  
 121 
$$\langle x, y \rangle = \sum_{n=1}^m (x_n, y_n), \quad \|x\| = \left( \sum_{n=1}^m |x_n|^2 \right)^{1/2}, \quad x, y \in E_m;$$

122  
 123 where  $(\cdot, \cdot)$  and  $|\cdot|$  are the usual inner product and norm on  $\mathbb{R}^N$ . Then,  $E_m$  is a  $mN$   
 124 dimensional Hilbert space.  
 125

126 We consider the linear operator  $A : E_m \rightarrow E_m$  defined by

127  
 128 
$$\langle Ax, y \rangle = \sum_{n=1}^m (\Delta x_n, \Delta y_n), \quad x, y \in E_m.$$

129  
 130 Then,  $A$  is semi-positive definite. The kernel space of  $A$  is

131  
 132 
$$W = \{x = \{x_n\}_{n \in \mathbb{Z}} : x_n = v \in \mathbb{R}^N\},$$

133  
 134 which is isomorphic to  $\mathbb{R}^N$ . Let  $Y = W^\perp$ , then  $E_m = Y \oplus W$ ,  $\dim Y = (m - 1)N$  and  $A$  is  
 135 positive definite on  $Y$ . It has been computed in [13] (Page 1017) that the smallest positive  
 136 eigenvalue of  $A$  is  
 137

138  
 139 
$$\lambda_{\min} = 2 \left( 1 - \cos \frac{2\pi}{m} \right).$$

140  
 141 Hence, for  $x \in Y$ , we have

142  
 143 
$$\sum_{n=1}^m |\Delta x_n|^2 = \langle Ax, x \rangle \geq \lambda_{\min} \|x\|^2. \tag{2.1}$$
  
 144  
 145  
 146  
 147

We define a functional  $\Phi : E_m \rightarrow \mathbb{R}$ ,

$$\Phi(x) = \sum_{n=1}^m \left[ \frac{1}{p} |\Delta x_n|^p - F(n, x_{n+1}, x_n) \right].$$

By a direct computation, we see that the critical points of  $\Phi$  are exactly the  $m$ -periodic solutions of (1.1), see [5] for the scalar case  $N = 1$ . Therefore to prove our theorems, it suffices to find critical points of  $\Phi$ . To this purpose, we need the following three critical points theorem and Clark’s theorem.

**PROPOSITION 2.1.** ([3,9]). *Let  $E$  be a Banach space,  $\Phi \in C^1(E, \mathbb{R})$  satisfies the Palais–Smale (PS) condition and is bounded from below. Suppose  $\Phi$  has a local linking at the origin 0, namely, there are a decomposition  $E = Y \oplus W$  and a positive real number  $\rho > 0$  such that  $k = \dim Y < \infty$ ,*

$$\begin{aligned} \Phi(x) < \Phi(0) \quad \text{for } x \in Y, \quad 0 < \|x\| \leq \rho, \quad \Phi(x) \geq \Phi(0) \\ \text{for } x \in W, \quad \|x\| \leq \rho; \end{aligned} \tag{2.2}$$

then  $\Phi$  has at least three critical points.

Recall that  $\Phi$  satisfies the (PS) condition, if any sequence  $\{x^{(i)}\}$  such that  $\{\Phi(x^{(i)})\}$  is bounded and  $\Phi'(x^{(i)}) \rightarrow 0$ , has a convergent subsequence.

**PROPOSITION 2.2.** ([6, 11] (THEOREM 9.1)). *Let  $E$  be a Banach space and  $\Phi \in C^1(E, \mathbb{R})$  be an even functional satisfying the (PS) condition and  $\Phi(0) = 0$ . Assume that  $\Phi$  is bounded from below and there are  $\rho > 0$  and a  $k$ -dimensional linear subspace  $Y$  of  $E$  such that*

$$\sup_{x \in Y, \|x\| = \rho} \Phi(x) < 0,$$

then  $\Phi$  possesses at least  $k$  pairs of critical points.

Note that these critical points are non-zero, because the values of  $\Phi$  over these points are negative, see the proof of [11, Theorem 9.1] for details.

Now, we recall some concept from the Morse theory, the reader is referred to [4,10] for more details. Let  $\Phi$  be a  $C^1$ -functional defined on a Banach space  $E$ , then the  $q$ th critical group of  $\Phi$  at an isolated critical point  $x$  with  $\Phi(x) = c$  is defined by

$$C_q(\Phi, x) = H_q(\Phi_c, \Phi_c \setminus \{x\}), \quad q \in \mathbb{N} := \{0, 1, 2, \dots\},$$

where  $H_*$  is the singular relative homology with coefficients in an Abelian group  $\mathcal{G}$  and  $\Phi_c = \Phi^{-1}(-\infty, c]$ . In the next section, we will use critical groups to distinguish critical points.

*Example 2.3.* Let  $\Phi \in C^1(\mathbb{R}^n, \mathbb{R})$ , if  $x$  is a local minimizer of  $\Phi$ , then

$$C_q(\Phi, x) \cong \delta_{q,0} \mathcal{G} := \begin{cases} \mathcal{G}, & q = 0, \\ 0, & q \neq 0. \end{cases}$$

If  $x$  is a local maximizer of  $\Phi$ , then  $C_q(\Phi, x) \cong \delta_{q,n}\mathcal{G}$ . These results can be easily obtained from the definition of critical groups and homology theory.

*Example 2.4.* Assume that  $\Phi \in C^1(E, \mathbb{R})$  satisfies the (PS) condition,  $\Phi$  has only finitely many critical points. If there exist  $\rho > 0$  and  $e \in E$  such that  $\|e\| > \rho$  and

$$\inf_{\|y\|=\rho} \Phi(y) > \max\{\Phi(0), \Phi(e)\},$$

then  $\Phi$  has a critical point  $x$  such that  $C_1(\Phi, x) \neq 0$ . In this setting, the existence of critical point is the well-known mountain pass lemma [1]. For the information about the critical groups, see [4,10].

### 3. Proof of the theorems

As the first step, we show that  $\Phi$  is anti-coercive.

LEMMA 3.1. *If  $(F_\infty)$  holds, then  $\Phi(x) \rightarrow -\infty$  as  $\|x\| \rightarrow \infty$ .*

*Proof.* By the definition of  $c$  in (1.2), we see that for  $a, b > 0$ , we have

$$\frac{a^p + b^p}{(a + b)^p} = \left(\frac{a}{a + b}\right)^p + \left(\frac{b}{a + b}\right)^p \geq c^{-1}, \quad \text{that is } (a + b)^p \leq c(a^p + b^p).$$

Noting that  $x_{n+m} = x_n$  for all  $n \in \mathbb{Z}$ , hence

$$\sum_{n=1}^m |x_{n+1}|^p = \sum_{n=1}^m |x_n|^p.$$

By  $(F_\infty)$ , we obtain

$$\begin{aligned} \Phi(x) &= \sum_{n=1}^m \left[ \frac{1}{p} |x_{n+1} - x_n|^p - F(n, x_{n+1}, x_n) \right] \\ &\leq \frac{1}{p} \sum_{n=1}^m (|x_{n+1}| + |x_n|)^p - \sum_{n=1}^m F(n, x_{n+1}, x_n) \\ &\leq \frac{c}{p} \sum_{n=1}^m (|x_{n+1}|^p + |x_n|^p) - \sum_{n=1}^m (a_1(|x_{n+1}|^p + |x_n|^p) - a_2) \\ &= \frac{2c}{p} \sum_{n=1}^m |x_n|^p - 2a_1 \sum_{n=1}^m |x_n|^p + ma_2 \rightarrow -\infty, \end{aligned}$$

as  $\|x\| \rightarrow \infty$ , because  $a_1 > p^{-1}c$ . □

Before giving the proof of our theorems, for any  $p > 1$ , we consider the  $p$ -norm  $|\cdot|_p$  on  $\mathbb{R}^m$ , namely, for  $v = (v_1, \dots, v_m) \in \mathbb{R}^m$ , we set

$$|v|_p = \left( \sum_{n=1}^m |v_n|^p \right)^{1/p}.$$

Since  $\dim \mathbb{R}^m < \infty$ , the two norms  $|\cdot|_2$  and  $|\cdot|_p$  are equivalent. Let  $c_1 > 0$  and  $c_2 > 0$  be the optimal constants such that

$$c_1|v|_2 \leq |v|_p \leq c_2|v|_2, \quad v \in \mathbb{R}^m. \tag{3.1}$$

*Proof.* [Proof of Theorem 1.1] Choose a positive number

$$\varepsilon < \frac{c_1^p}{2pc_2^p} \lambda_{\min}^{p/2}.$$

By  $(F_0)$ , there exists  $\rho \in (0, \eta)$  such that

$$F(n, u, v) \leq \varepsilon(|u|^p + |v|^p), \quad \text{for } |u| + |v| \leq 2\rho. \tag{3.2}$$

Now, if  $x = \{x_n\} \in Y$ ,  $0 < \|x\| \leq \rho$ , then  $|x_n| \leq \rho$  for all  $n \in \mathbb{Z}$ . Using (3.1) with

$$v = (|\Delta x_1|, \dots, |\Delta x_m|),$$

and (2.1), as well as (3.2), we obtain

$$\begin{aligned} \Phi(x) &= \sum_{n=1}^m \left[ \frac{1}{p} |\Delta x_n|^p - F(n, x_{n+1}, x_n) \right] \\ &\geq \frac{1}{p} \left[ \left( \sum_{n=1}^m |\Delta x_n|^p \right)^{1/p} \right]^p - \varepsilon \sum_{n=1}^m (|x_{n+1}|^p + |x_n|^p) \\ &\geq \frac{1}{p} c_1^p \left( \sum_{n=1}^m |\Delta x_n|^2 \right)^{p/2} - 2\varepsilon \sum_{n=1}^m |x_n|^p \\ &\geq \frac{c_1^p}{p} \lambda_{\min}^{p/2} \|x\|^p - 2\varepsilon c_2^p \|x\|^p = \left( \frac{c_1^p}{p} \lambda_{\min}^{p/2} - 2\varepsilon c_2^p \right) \|x\|^p > 0. \end{aligned} \tag{3.3}$$

On the other hand, if  $x \in W$ ,  $\|x\| \leq \rho$ , then for all  $n \in \mathbb{Z}$ , we have

$$|x_{n+1}| + |x_n| < 2\eta,$$

and  $\Delta x_n = 0$ . Thus, by our assumption  $(F_0)$ , we obtain

$$\Phi(x) = - \sum_{n=1}^m F(n, x_{n+1}, x_n) \leq 0. \tag{3.4}$$

Since  $F(n, 0, 0) = 0$ , so  $\Phi(0) = 0$ . It follows from (3.3) and (3.4) that  $-\Phi$  has a local linking at the origin 0 with respect to the decomposition  $E_m = Y \oplus W$ .

Since  $\dim E_m < \infty$ , by Lemma 3.1, it is easy to see that  $-\Phi$  is bounded from below and satisfies the (PS) condition. Applying Proposition 2.1,  $-\Phi$  has at least three critical points. Therefore,  $\Phi$  has two non-zero critical points, which are non-trivial  $m$ -periodic solutions to our problem (1.1). □

*Remark 1.* In [5], after obtaining an estimate similar to (3.3), in order to apply the linking theorem, some tedious estimates are involved, and the global condition  $F(n, u, v) \geq 0$  for all  $(n, u, v)$  is needed for verifying that  $\Phi(x) \leq 0$  for all  $x \in \partial Q$ ; here  $Q$  is defined in [5] (equation (3.20)). Our argument above does not need this global condition, and simplifies the proof considerably.

*Proof.* [Proof of Theorem 1.2]. By Lemma 3.1 and the fact that  $\dim E_m < \infty$ , we see that  $\Phi$  satisfies (PS) and there exists  $x^1 \in E_m$  such that

$$\Phi(x^1) = \max_{x \in E_m} \Phi(x).$$

For  $x \in E_m$  with  $\|x\| \leq \eta$ , we have  $|x_{n+1}| + |x_n| \leq 2\eta$ . Therefore, by our assumptions, we obtain

$$\Phi(x) \geq -\sum_{n=1}^m F(n, x_{n+1}, x_n) \geq 0 = \Phi(0).$$

So, 0 is a local minimizer of  $\Phi$ . If either 0 or  $x^1$  is not a isolated critical point of  $\Phi$ , then  $\Phi$  has infinitely many critical points, which are all  $m$ -periodic solutions of (1.1).

Therefore, we may assume that both 0 and  $x^1$  are isolated critical points of  $\Phi$ . Now, by Example 2.3, we obtain

$$C_q(\Phi, 0) \cong \delta_{q,0}\mathcal{G}, \quad C_q(\Phi, x^1) \cong \delta_{q,mN}\mathcal{G}. \tag{3.5}$$

By Lemma 3.1, we see that  $\Phi(x) \rightarrow -\infty$  as  $\|x\| \rightarrow \infty$ . Note that 0 is an isolated local minimizer of  $\Phi$ , it is then easy to see that there exist  $\rho > 0$  and  $e \in E$  such that

$$\inf_{\|y\|=\rho} \Phi(y) > \max\{\Phi(0), \Phi(e)\}.$$

By Example 2.4,  $\Phi$  has a critical point  $x^2$  such that  $C_1(\Phi, x^2) \neq 0$ . Compare this with (3.5), we see that  $x^1$  and  $x^2$  are non-zero critical points of  $\Phi$ . Hence, they are non-trivial  $m$ -periodic solutions of the problem (1.1). □

*Proof.* [Proof of Theorem 1.3]. If  $F(n, u, v) = F(n, -u, -v)$  for all  $(n, u, v)$ , then  $\Phi$  is an even functional. We know that  $\Phi(0) = 0$ ,  $-\Phi$  is bounded from below and satisfies the (PS) condition. Using (3.3), we see that

$$\sup_{x \in Y, \|x\|=\rho} (-\Phi)(x) \leq \left( 2\epsilon c_2^p - \frac{c_1^p}{p} \lambda_{\min}^{p/2} \right) \rho^p < 0.$$

Since  $\dim Y = (m - 1)N$ , the desired result follows from Proposition 2.2. □

344 *Remark 2.* By our argument above, in our Theorems 1.1 and 1.3, we may replace the limit  
 345 (1.3) with the weaker condition (3.2).  
 346

## 347 **Q1 Acknowledgements**

348 The author is very grateful to the anonymous referee for the careful reading of the manuscript and for  
 349 pointing out some errors in the original manuscript. This work was supported by the National Natural  
 350 Science Foundation of China (10601041).  
 351

## 352 **References**

- 353 [1] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and*  
 354 *applications*, J. Funct. Anal. 14 (1973), pp. 349–381.
- 355 [2] H.H. Bin, J.S. Yu, and Z.M. Guo, *Nontrivial periodic solutions for asymptotically linear*  
 356 *resonant difference proble*, J. Math. Anal. Appl. 322 (2006), pp. 477–488.
- 357 [3] H. Brezis and L. Nirenberg, *Remarks on finding critical points*, Comm. Pure Appl. Math. 44  
 358 (1991), pp. 939–963.
- 359 [4] K.C. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problem*, Birkhäuser,  
 Boston, MA, 1993.
- 360 [5] P. Chen and H. Fang, *Existence of periodic and subharmonic solutions for second-order*  
 361 *p-Laplacian difference equations*, Adv. Difference Equ. 2007 (2007), art. id. 42530.
- 362 [6] D.C. Clark, *A variant of the Ljusternik–Schnirelmann theory*, Indiana Univ. Math. J. 22 (1972),  
 363 pp. 65–74.
- 364 [7] Z.M. Guo and J.S. Yu, *Existence of periodic and subharmonic solutions for second-order*  
 365 *superlinear difference equations*, Sci. China Ser. A 46 (2003), pp. 506–515.
- 366 [8] Z.M. Guo and J.S. Yu, *The existence of periodic and subharmonic solutions to subquadratic*  
 367 *second order difference equations*, J. London Math. Soc. 68 (2003), pp. 419–430.
- 368 [9] J.Q. Liu and S.J. Li, *Existence theorems of multiple critical points and their applications*,  
 369 Kexue Tongbao 17 (1984), pp. 1025–1027.
- 370 [10] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, Berlin,  
 371 1989.
- 372 [11] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential*  
 373 *Equations*, Vol. 65, CBMS, American Mathematical Society, Providence, RI, 1986.
- 374 [12] D. Smets and M. Willem, *Solitary waves with prescribed speed on infinite lattices*, J. Funct.  
 375 Anal. 149 (1997), pp. 266–275.
- 376 [13] Z. Zhou, J.S. Yu, and Z.M. Guo, *Periodic solutions of higher-dimensional discrete systems*,  
 377 Proc. Roy. Soc. Edinburgh Sect. A 134 (2004), pp. 1013–1022.  
 378  
 379  
 380  
 381  
 382  
 383  
 384  
 385  
 386  
 387  
 388  
 389  
 390  
 391  
 392