

# Multiple periodic solutions for nonlinear difference systems involving the $p$ -Laplacian <sup>☆</sup>

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## Abstract

Using the three critical points theorem, the Clark's theorem and Morse theory, multiple periodic solutions for nonlinear difference systems involving the  $p$ -Laplacian are obtained by variational methods.

*Key words:* nonlinear difference systems; local linking; three critical points theorem; critical groups; Clark's theorem.  
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## 1. Introduction

Let  $m > 1$  be a fixed integer, we consider the existence of multiple  $m$ -periodic solutions for nonlinear difference systems of the form

$$\Delta(|\Delta x_{n-1}|^{p-2} \Delta x_{n-1}) + f(n, x_{n+1}, x_n, x_{n-1}) = 0, \quad x_{n+m} = x_n, \quad n \in \mathbb{Z}; \quad (1.1)$$

where  $p > 1$ ,  $\Delta$  is the forward difference operator defined by  $\Delta x_n = x_{n+1} - x_n$ ,  $f : \mathbb{Z} \times \mathbb{R}^{3N} \rightarrow \mathbb{R}^N$  is continuous and there exists  $F(\cdot, \cdot, \cdot) \in C^1(\mathbb{Z} \times \mathbb{R}^{2N}, \mathbb{R})$  such that

$$f(n, u, v, w) = F'_2(n-1, v, w) + F'_3(n, u, v);$$

here  $F'_i$  denotes the partial derivative of  $F$  with respect to the  $i$ -th variable. When  $p = 2$  and  $F(t, u, v)$  is independent of  $u$ , that is  $F(t, u, v) = G(t, v)$  for some  $G \in C^1(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$ , we have  $F'_2 = \partial_u F = 0$ . Hence

$$f(n, x_{n+1}, x_n, x_{n-1}) = F'_3(n, x_{n+1}, x_n) = \partial_v G(n, x_n) =: G'(n, x_n).$$

We see that the system in (1.1) reduces to the second order discrete Hamiltonian system

$$\Delta^2 x_{n-1} + G'(n, x_n) = 0,$$

which has been discussed in [2, 13].

We may consider the problem (1.1) as a discrete analogue of the following nonlinear functional differential equation involving the  $p$ -Laplacian:

$$(|x'|^{p-2} x')' + f(t, x(t+1), x(t), x(t-1)) = 0, \quad t \in \mathbb{R}.$$

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This kind of equation arise in the study of the existence of solitary waves of lattice differential equations, see [12] and the references therein.

To state our main results, we denote by  $c$  the real number such that

$$c^{-1} = \min_{x,y>0, x+y=1} (x^p + y^p) > 0. \quad (1.2)$$

We assume that the function  $F$  satisfies the following conditions:

$(F_\star)$   $F \in C^1(\mathbb{Z} \times \mathbb{R}^{2N}, \mathbb{R})$ ,  $F(n, 0, 0) = 0$  and

$$F(n, u, v) = F(n + m, u, v), \quad \text{for all } (n, u, v) \in \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N;$$

$(F_0)$  there exists some  $\eta > 0$  such that  $F(n, u, v) \geq 0$  for  $|u| + |v| \leq 2\eta$ , and

$$\lim_{|u|+|v|\rightarrow 0} \frac{F(n, u, v)}{|u|^p + |v|^p} = 0, \quad \text{uniformly in } n \in \mathbb{Z}; \quad (1.3)$$

$(F_\infty)$  there exist  $a_1 > p^{-1}c$  and  $a_2 > 0$  such that

$$F(n, u, v) \geq a_1(|u|^p + |v|^p) - a_2, \quad \text{for all } (n, u, v) \in \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N. \quad (1.4)$$

It follows from (1.3) that  $f(n, 0, 0, 0) = 0$ , therefore  $x_n = 0$  is a trivial  $m$ -periodic solution of the problem (1.1). Therefore we want to find nontrivial solutions. The main result of this paper is the following theorems.

**Theorem 1.1.** *Suppose that  $(F_\star)$ ,  $(F_0)$  and  $(F_\infty)$  hold, then the problem (1.1) has at least two nontrivial  $m$ -periodic solutions.*

In a recent paper Chen-Fang [5], the scalar case  $N = 1$  has been considered by variational methods. Under  $(F_\star)$ , (1.3) and some conditions slightly stronger than  $(F_\infty)$ , as well as  $F(n, u, v) \geq 0$  for all  $(n, u, v) \in \mathbb{Z} \times \mathbb{R} \times \mathbb{R}$ , they obtained two nonzero periodic solutions by the linking theorem [11, Theorem 5.3]. In that paper,  $F(n, u, v) \geq 0$  is a global requirement, which is crucial for applying the linking theorem.

The novelty of our Theorem 1.1 is that, we only require  $F(n, u, v) \geq 0$  for small  $|u|$  and  $|v|$ . In this case the linking theorem is no longer applicable. Our proof of Theorem 1.1 is also based on variational methods. However, instead of the linking theorem, we will use the three critical points theorem of Liu-Li [9] and Brezis-Nirenberg [3]. It turns out that this approach is considerably simpler.

We may also consider the case that  $F(n, u, v) \leq 0$  for small  $|u|$  and  $|v|$ . In this situation we have the following result, which is obtained by Morse theory.

**Theorem 1.2.** *Suppose that  $(F_\star)$ ,  $(F_\infty)$  hold. If there exists  $\eta > 0$  such that  $F(n, u, v) \leq 0$  for  $|u| + |v| \leq 2\eta$ , then the problem (1.1) has at least two nontrivial  $m$ -periodic solutions.*

It turns out again that  $x_n = 0$  is a trivial  $m$ -periodic solution of the problem (1.1). We emphasize that, in Theorem 1.2 we do not require the limit condition (1.3). In our next result, we consider the case that  $F(n, u, v)$  is even in  $(u, v)$ , then we can obtain more solutions.

**Theorem 1.3.** *Suppose that  $F(n, u, v) = F(n, -u, -v)$  for all  $(n, u, v) \in \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N$ . If  $F$  satisfies  $(F_\star)$ ,  $(F_0)$  and  $(F_\infty)$ , then the problem (1.1) has at least  $(m - 1)N$  pairs of nontrivial  $m$ -periodic solutions.*

This symmetric case has not been considered in [5]. The proof of this theorem is based on variational methods and the Clark's theorem [6].

The variational methods and critical point theory has been extensively applied to differential equations, see Rabinowitz [11] and Mawhin-Willem [10] for an excellent survey. Since the appearance of [8], in recent years this approach has also been used to study difference equations by many authors, see for example [2, 5, 7, 13].

## 2. Variational framework

Following [2], for our fixed integer  $m > 1$ , we define the linear operations on

$$E_m = \left\{ x = \{x_n\}_{n \in \mathbb{Z}} : x_n \in \mathbb{R}^N, x_{n+m} = x_n, n \in \mathbb{Z} \right\}$$

in an obvious way, and then define the inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$  on  $E_m$  as follows:

$$\langle x, y \rangle = \sum_{n=1}^m (x_n, y_n), \quad \|x\| = \left( \sum_{n=1}^m |x_n|^2 \right)^{1/2}, \quad x, y \in E_m;$$

where  $(\cdot, \cdot)$  and  $|\cdot|$  are the usual inner product and norm on  $\mathbb{R}^N$ . Then  $E_m$  is a  $mN$  dimensional Hilbert space.

We consider the linear operator  $A : E_m \rightarrow E_m$  defined by

$$\langle Ax, y \rangle = \sum_{n=1}^m (\Delta x_n, \Delta y_n), \quad x, y \in E_m.$$

Then  $A$  is semi positive definite. The kernel space of  $A$  is

$$W = \left\{ x = \{x_n\}_{n \in \mathbb{Z}} : x_n = v \in \mathbb{R}^N \right\},$$

which is isomorphic to  $\mathbb{R}^N$ . Let  $Y = W^\perp$ , then  $E_m = Y \oplus W$ ,  $\dim Y = (m-1)N$ , and  $A$  is positive definite on  $Y$ . It has been computed in [13, Page 1017] that the smallest positive eigenvalue of  $A$  is

$$\lambda_{\min} = 2 \left( 1 - \cos \frac{2\pi}{m} \right).$$

Hence, for  $x \in Y$  we have

$$\sum_{n=1}^m |\Delta x_n|^2 = \langle Ax, x \rangle \geq \lambda_{\min} \|x\|^2. \quad (2.1)$$

We define a functional  $\Phi : E_m \rightarrow \mathbb{R}$ ,

$$\Phi(x) = \sum_{n=1}^m \left[ \frac{1}{p} |\Delta x_n|^p - F(n, x_{n+1}, x_n) \right].$$

By a direct computation, we see that the critical points of  $\Phi$  are exactly the  $m$ -periodic solutions of (1.1), see [5] for the scalar case  $N = 1$ . Therefore to prove our theorems it suffices to find critical points of  $\Phi$ . To this purpose, we need the following three critical points theorem and the Clark's theorem.

**Proposition 2.1** ([9, 3]). *Let  $E$  be a Banach space,  $\Phi \in C^1(E, \mathbb{R})$  satisfies the Palais-Smale (PS) condition and is bounded from below. Suppose  $\Phi$  has a local linking at the origin 0, namely,*

there are a decomposition  $E = Y \oplus W$  and a positive real number  $\rho > 0$  such that  $k = \dim Y < \infty$ ,

$$\Phi(x) < \Phi(0) \text{ for } x \in Y, 0 < \|x\| \leq \rho, \quad \Phi(x) \geq \Phi(0) \text{ for } x \in W, \|x\| \leq \rho; \quad (2.2)$$

then  $\Phi$  has at least three critical points.

Recall that  $\Phi$  satisfies the (PS) condition, if any sequence  $\{x^{(i)}\}$  such that  $\{\Phi(x^{(i)})\}$  is bounded and  $\Phi'(x^{(i)}) \rightarrow 0$ , has a convergent subsequence.

**Proposition 2.2** ([6], [11, Theorem 9.1]). *Let  $E$  be a Banach space and  $\Phi \in C^1(E, \mathbb{R})$  be an even functional satisfying the (PS) condition and  $\Phi(0) = 0$ . Assume that  $\Phi$  is bounded from below and there are  $\rho > 0$  and a  $k$ -dimensional linear subspace  $Y$  of  $E$  such that*

$$\sup_{x \in Y, \|x\| = \rho} \Phi(x) < 0,$$

then  $\Phi$  possesses at least  $k$  pairs of critical points.

Note that these critical points are nonzero, because the values of  $\Phi$  over these points are negative, see the proof of [11, Theorem 9.1] for the details.

Now we recall some concept from Morse theory, the reader is referred to [4, 10] for more details. Let  $\Phi$  be a  $C^1$ -functional defined on a Banach space  $E$ , then the  $q$ -th critical group of  $\Phi$  at an isolated critical point  $x$  with  $\Phi(x) = c$  is defined by

$$C_q(\Phi, x) = H_q(\Phi_c, \Phi_c \setminus \{x\}), \quad q \in \mathbb{N} := \{0, 1, 2, \dots\};$$

where  $H_*$  is the singular relative homology with coefficients in an Abelian group  $\mathcal{G}$  and  $\Phi_c = \Phi^{-1}(-\infty, c]$ . In the next section, we will use critical groups to distinguish critical points.

*Example 2.3.* Let  $\Phi \in C^1(\mathbb{R}^n, \mathbb{R})$ , if  $x$  is a local minimizer of  $\Phi$ , then

$$C_q(\Phi, x) \cong \delta_{q,0} \mathcal{G} := \begin{cases} \mathcal{G}, & q = 0, \\ 0, & q \neq 0. \end{cases}$$

If  $x$  is a local maximizer of  $\Phi$ , then  $C_q(\Phi, x) \cong \delta_{q,n} \mathcal{G}$ . These results can be easily obtained from the definition of critical groups and homology theory.

*Example 2.4.* Assume that  $\Phi \in C^1(E, \mathbb{R})$  satisfies the (PS) condition,  $\Phi$  has only finitely many critical points. If there exist  $\rho > 0$  and  $e \in E$  such that  $\|e\| > \rho$  and

$$\inf_{\|y\| = \rho} \Phi(y) > \max \{\Phi(0), \Phi(e)\},$$

then  $\Phi$  has a critical point  $x$  such that  $C_1(\Phi, x) \neq 0$ . In this setting, the existence of critical point is the well known mountain pass lemma [1]. For the information about the critical groups, see [4, 10].

### 3. Proof of the theorems

As the first step, we show that  $\Phi$  is anti-coercive.

**Lemma 3.1.** *If  $(F_\infty)$  holds, then  $\Phi(x) \rightarrow -\infty$  as  $\|x\| \rightarrow \infty$ .*

*Proof.* By the definition of  $c$  in (1.2), we see that for  $a, b > 0$ , we have

$$\frac{a^p + b^p}{(a + b)^p} = \left(\frac{a}{a + b}\right)^p + \left(\frac{b}{a + b}\right)^p \geq c^{-1}, \quad \text{that is } (a + b)^p \leq c(a^p + b^p).$$

Noting that  $x_{n+m} = x_n$  for all  $n \in \mathbb{Z}$ , hence

$$\sum_{n=1}^m |x_{n+1}|^p = \sum_{n=1}^m |x_n|^p.$$

By  $(F_\infty)$  we obtain

$$\begin{aligned} \Phi(x) &= \sum_{n=1}^m \left[ \frac{1}{p} |x_{n+1} - x_n|^p - F(n, x_{n+1}, x_n) \right] \\ &\leq \frac{1}{p} \sum_{n=1}^m (|x_{n+1}| + |x_n|)^p - \sum_{n=1}^m F(n, x_{n+1}, x_n) \\ &\leq \frac{c}{p} \sum_{n=1}^m (|x_{n+1}|^p + |x_n|^p) - \sum_{n=1}^m (a_1(|x_{n+1}|^p + |x_n|^p) - a_2) \\ &= \frac{2c}{p} \sum_{n=1}^m |x_n|^p - 2a_1 \sum_{n=1}^m |x_n|^p + ma_2 \rightarrow -\infty \end{aligned}$$

as  $\|x\| \rightarrow \infty$ , because  $a_1 > p^{-1}c$ . □

Before giving the proof of our theorems, for any  $p > 1$  we consider the  $p$ -norm  $|\cdot|_p$  on  $\mathbb{R}^m$ , namely, for  $v = (v_1, \dots, v_m) \in \mathbb{R}^m$  we set

$$|v|_p = \left( \sum_{n=1}^m |v_n|^p \right)^{1/p}.$$

Since  $\dim \mathbb{R}^m < \infty$ , the two norms  $|\cdot|_2$  and  $|\cdot|_p$  are equivalent. Let  $c_1 > 0$  and  $c_2 > 0$  be the optimal constants such that

$$c_1 |v|_2 \leq |v|_p \leq c_2 |v|_2, \quad v \in \mathbb{R}^m. \quad (3.1)$$

*Proof of Theorem 1.1.* Choose a positive number

$$\varepsilon < \frac{c_1^p}{2pc_2^p} \lambda_{\min}^{p/2}.$$

By  $(F_0)$ , there exists  $\rho \in (0, \eta)$  such that

$$F(n, u, v) \leq \varepsilon(|u|^p + |v|^p), \quad \text{for } |u| + |v| \leq 2\rho. \quad (3.2)$$

Now, if  $x = \{x_n\} \in Y$ ,  $0 < \|x\| \leq \rho$ , then  $|x_n| \leq \rho$  for all  $n \in \mathbb{Z}$ . Using (3.1) with

$$v = (|\Delta x_1|, \dots, |\Delta x_m|)$$

and (2.1), as well as (3.2), we obtain

$$\Phi(x) = \sum_{n=1}^m \left[ \frac{1}{p} |\Delta x_n|^p - F(n, x_{n+1}, x_n) \right]$$

$$\begin{aligned}
&\geq \frac{1}{p} \left[ \left( \sum_{n=1}^m |\Delta x_n|^p \right)^{1/p} \right]^p - \varepsilon \sum_{n=1}^m (|x_{n+1}|^p + |x_n|^p) \\
&\geq \frac{1}{p} c_1^p \left( \sum_{n=1}^m |\Delta x_n|^2 \right)^{p/2} - 2\varepsilon \sum_{n=1}^m |x_n|^p \\
&\geq \frac{c_1^p}{p} \lambda_{\min}^{p/2} \|x\|^p - 2\varepsilon c_2^p \|x\|^p = \left( \frac{c_1^p}{p} \lambda_{\min}^{p/2} - 2\varepsilon c_2^p \right) \|x\|^p > 0. \tag{3.3}
\end{aligned}$$

On the other hand, if  $x \in W$ ,  $\|x\| \leq \rho$ , then for all  $n \in \mathbb{Z}$  we have

$$|x_{n+1}| + |x_n| < 2\eta$$

and  $\Delta x_n = 0$ . Thus by our assumption  $(F_0)$ , we obtain

$$\Phi(x) = - \sum_{n=1}^m F(n, x_{n+1}, x_n) \leq 0. \tag{3.4}$$

Since  $F(n, 0, 0) = 0$ , so  $\Phi(0) = 0$ . It follows from (3.3) and (3.4) that  $-\Phi$  has a local linking at the origin 0 with respect to the decomposition  $E_m = Y \oplus W$ .

Since  $\dim E_m < \infty$ , by Lemma 3.1 it is easy to see that  $-\Phi$  is bounded from below and satisfies the  $(PS)$  condition. Applying Proposition 2.1,  $-\Phi$  has at least three critical points. Therefore,  $\Phi$  has two nonzero critical points, which are nontrivial  $m$ -periodic solutions to our problem (1.1).  $\square$

*Remark 3.2.* In [5], after obtaining an estimate similar to (3.3), in order to apply the linking theorem, some tedious estimates are involved, and the global condition  $F(n, u, v) \geq 0$  for all  $(n, u, v)$  is needed for verifying that  $\Phi(x) \leq 0$  for all  $x \in \partial Q$ ; here  $Q$  is defined in [5, Eq. (3.20)]. Our argument above does not need this global condition, and simplifies the proof considerably.

*Proof of Theorem 1.2.* By Lemma 3.1 and the fact that  $\dim E_m < \infty$ , we see that  $\Phi$  satisfies  $(PS)$  and there exists  $x^1 \in E_m$  such that

$$\Phi(x^1) = \max_{x \in E_m} \Phi(x).$$

For  $x \in E_m$  with  $\|x\| \leq \eta$ , we have  $|x_{n+1}| + |x_n| \leq 2\eta$ . Therefore, by our assumptions we obtain

$$\Phi(x) \geq - \sum_{n=1}^m F(n, x_{n+1}, x_n) \geq 0 = \Phi(0).$$

So 0 is a local minimizer of  $\Phi$ . If either 0 or  $x^1$  is not isolated critical point of  $\Phi$ , then  $\Phi$  has infinitely many critical points, which are all  $m$ -periodic solutions of (1.1).

Therefore we may assume that both 0 and  $x^1$  are isolated critical point of  $\Phi$ . Now by Example 2.3 we obtain

$$C_q(\Phi, 0) \cong \delta_{q,0} \mathcal{G}, \quad C_q(\Phi, x^1) \cong \delta_{q,mN} \mathcal{G}. \tag{3.5}$$

By Lemma 3.1 we see that  $\Phi(x) \rightarrow -\infty$  as  $\|x\| \rightarrow \infty$ . Note that 0 is an isolated local minimizer of  $\Phi$ , it is then easy to see that there exist  $\rho > 0$  and  $e \in E$  such that

$$\inf_{\|y\|=\rho} \Phi(y) > \max \{ \Phi(0), \Phi(e) \}.$$

By Example 2.4,  $\Phi$  has a critical point  $x^2$  such that  $C_1(\Phi, x^2) \neq 0$ . Compare this with (3.5), we see that  $x^1$  and  $x^2$  are nonzero critical points of  $\Phi$ . Hence they are nontrivial  $m$ -periodic solutions of the problem (1.1).  $\square$

*Proof of Theorem 1.3.* If  $F(n, u, v) = F(n, -u, -v)$  for all  $(n, u, v)$ , then  $\Phi$  is an even functional. We know that  $\Phi(0) = 0$ ,  $-\Phi$  is bounded from below and satisfies the  $(PS)$  condition. Using (3.3) we see that

$$\sup_{x \in Y, \|x\|=\rho} (-\Phi)(x) \leq \left( 2\epsilon c_2^p - \frac{c_1^p}{p} \lambda_{\min}^{p/2} \right) \rho^p < 0.$$

Since  $\dim Y = (m - 1)N$ , the desired result follows from Proposition 2.2.  $\square$

*Remark 3.3.* By our argument above, in our Theorems 1.1 and 1.3, we may replace the limit (1.3) with the weaker condition (3.2).

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