Nontrivial solutions for elliptic resonant problems [☆]

Shibo Liu [a,*]

[a] *Department of Mathematics, Xiamen University,*
Xiamen 361005, China

Abstract

Nontrivial solutions for elliptic resonant problems are obtained via Morse theory. To compute the critical groups at infinity of the relevant functional, we propose a new approach by combining the homotopy, reduction methods and the Alexander Duality Theorem.

Key words: Elliptic resonant problems, Critical groups, Lyapunov-Schmidt reduction, Alexander Duality Theorem

MSC: 35J65, 58E05

---

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We consider the following semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = p(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies

$$|p(x,t_1) - p(x,t_2)| \le \Lambda\,|t_1 - t_2| \tag{1.2}$$

for some $\Lambda > 0$. Assume that $p(x,0) = 0$, then (1.1) admits a trivial solution $u = \mathbf{0}$. Therefore we are concerned on the existence of nontrivial solutions.

Let $\lambda_1 < \lambda_2 \le \cdots \le \lambda_m < \lambda_{m+1} \le \cdots$ denote the eigenvalues of $(-\Delta, H_0^1(\Omega))$, we consider the following conditions of p:

(p_0) there exists $\rho > 0$ such that for $|t| \le \rho$

$$\frac{1}{2}\lambda_m t^2 \le P(x,t) \le \frac{1}{2}\lambda_{m+1} t^2, \qquad \text{where } P(x,t) = \int_0^t p(x,s)\,\mathrm{d}s,$$

(p_-) there exist $\varepsilon, R > 0$ such that for $|t| \ge R$

$$\lambda_{\ell-1} + \varepsilon \le \frac{p(x,t)}{t} \le \lambda_\ell, \qquad \lim_{|t|\to\infty} \frac{1}{|t|}\left(P(x,t) - \frac{\lambda_\ell}{2}t^2\right) = -\infty. \tag{1.3}$$

[☆] Supported by National Natural Science Foundation of China (10601041).

[*] Current address: Department of Mathematics, Shantou University, Shantou 515063, China.
Email address: `liusb@stu.edu.cn` (Shibo Liu).

(p_+) there exist $\varepsilon, R > 0$ such that for $|t| \geq R$

$$\lambda_\ell \leq \frac{p(x,t)}{t} \leq \lambda_{\ell+1} - \varepsilon, \qquad \lim_{|t|\to\infty} \frac{1}{|t|}\left(P(x,t) - \frac{\lambda_\ell}{2}t^2\right) = +\infty, \tag{1.4}$$

Note that the inequalities in $(p_\pm)$ characterize (1.1) as resonant from one side at the eigenvalue λ_ℓ. In particular, when the limit

$$\lim_{|t|\to\infty} \frac{p(x,t)}{t} = \lambda_\ell \tag{1.5}$$

exists, we say that the problem (1.1) is resonant at infinity. Such resonant problems have been extensively studied by many authors with various conditions imposed on the nonlinearity, see [2, 4, 9, 11, 16, 18]. Typically, many results in the literature assume that the nonlinearity $p(x,t) - \lambda_\ell t$ is bounded or grows slower than $|t|^\alpha$ for some $\alpha \in (0,1)$. For example, the authors of [9] considered the case that p satiefies

$$|p(x,t) - \lambda_\ell t| \leq c\left(1 + |t|^\alpha\right), \qquad \lim_{|t|\to\infty} \frac{1}{|t|^{2\alpha}}\left(P(x,t) - \frac{\lambda_\ell}{2}t^2\right) = \pm\infty \tag{1.6}$$

for some $\alpha \in [0,1)$, and obtained one nontrivial solution of (1.1). Note that if $\alpha = 0$, then (1.6) is the standard Landesman–Lazer condition and has been considered in [4, 8].

In this paper, we study the case that $p(x,t) - \lambda_\ell t$ may not be bounded, not even grows slower than $|t|^\alpha$ for some $\alpha \in (0,1)$. We shall prove the following theorem.

Theorem 1.1. *Let p be a Carathéodory function on $\Omega \times \mathbb{R}$ that satisfies* (1.2). *Then the problem* (1.1) *has at least a nontrivial solution in one of the following cases:*

(i) *p satisfies (p_0) and (p_-) with $m \neq \ell - 1$.*
(ii) *p satisfies (p_0) and (p_+) with $m \neq \ell$.*

Let $X = H_0^1(\Omega)$ be the Sobolev space with inner product

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v \,\mathrm{d}x$$

and corresponding norm $\|u\| = \langle u, u\rangle^{1/2}$, we shall prove Theorem 1.1 by showing that, the functional $f : X \to \mathbb{R}$,

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega P(x,u)\,\mathrm{d}x$$

possess nonzero critical points. To do that, we shall apply the Morse theory (see the monographs [4, 15] for a systematic exploration). More precisely, we will obtain nonzero critical points of f by comparing the critical groups $C_*(f, \mathbf{0})$ of f at zero and $C_*(f,\infty)$ at infinity (see Proposition 2.1 below). Therefore, the computation of critical groups is essential. The following theorem is devoted to this issue. Since its proof (in particular, Case (ii)) illustrates a new approach to compute $C_*(f,\infty)$, we recognize it as another main result of this paper.

Theorem 1.2. *Let p be a Carathéodory function on $\Omega \times \mathbb{R}$ that satisfies* (1.2).

(i) *If p satisfies (p_-), then the functional f satisfies the Palais-Smale (PS) condition, and $C_k(f,\infty) \cong \delta_{k,\ell-1}\mathscr{G}$.*
(ii) *If p satisfies (p_+), then the functional f satisfies the Palais-Smale (PS) condition, and $C_k(f,\infty) \cong \delta_{k,\ell}\mathscr{G}$.*

Remark 1.3. This Theorem improves Li-Perera-Su [10, Proposition 1.3]: in Case (i), instead of (1.3) they assumed the stronger condition: for $|t| \geq R$,

$$\lambda_{\ell-1} + \varepsilon \leq \frac{p(x,t)}{t} \leq \lambda_\ell, \qquad P(x,t) \leq \frac{\lambda_\ell - \varepsilon}{2} t^2,$$

which can not be satisfied if (1.5) holds; in Case (ii), instead of (1.4) they assumed the stronger condition: for $|t| \geq R$

$$\lambda_\ell \leq \frac{p(x,t)}{t} \leq \lambda_{\ell+1} - \varepsilon, \qquad P(x,t) \geq \frac{\lambda_\ell + \varepsilon}{2} t^2,$$

which can not be satisfied if (1.5) holds, moreover, they only obtained $C_\ell(f,\infty) \neq 0$.

We will prove Theorem 1.2 in the next section. Our proof is based on the Homotopy Invariant Theorem for $C_*(f,\infty)$ established in [10]. However, this method can not be used directly in Case (ii) of Theorem 1.2. Therefore, we use the reduction method of [14] and then, apply the Alexander Duality Theorem. Obviously, our method can also be used in other situation.

In Section 3, we apply Theorem 1.2 and Morse theory to the functional f and give the proof of Theorem 1.1. Other multiplicity results for boundary value problems are also given in the section.

2. Computation of $C_*(f,\infty)$

Let f be a C^1-functional defined on a Hilbert space X, then the critical groups of f at an isolated critical point u with $f(u) = c$ are defined by

$$C_k(f,u) := H_k(f^c, f^c \setminus \{u\}), \qquad k \in \mathbb{N} = \{0,1,2,\cdots\},$$

where H_* is the singular relative homology with coefficients in an Abelian group $\mathscr{G}$ and $f^c = f^{-1}(-\infty, c]$. If f satisfies the (PS) condition and the critical values of f are bonded from below by some $\alpha > -\infty$, then the critical groups of f at infinity were introduced by Bartsch-Li [2] as

$$C_k(f,\infty) := H_k(X, f^\alpha), \qquad k \in \mathbb{N}. \tag{2.1}$$

Note that by the deformation lemma, the right-hand side of (2.1) does not depend on the choice of α.

Using these concepts, we have the following famous Morse inequality

$$\sum_{k=0}^{\infty} M_k t^k = \sum_{k=0}^{\infty} \beta_k t^k + (1+t)\, Q(t), \tag{2.2}$$

where Q is a formal series with coefficients in $\mathbb{N}$,

$$M_k = \sum\nolimits_{f'(u)=0} \operatorname{rank} C_k(f,u), \qquad \beta_k = \operatorname{rank} C_k(f,\infty).$$

The critical groups and Morse inequality are very useful in studying the existence and multiplicity of critical points: we can distinguish critical points using critical groups and discover unknown critical points using the Morse inequality.

The reader is referred to [4, 15] for more details on Morse theory. Here we state a corollary of the Morse inequality (2.2), which is sufficient for proving Theorem 1.1.

Proposition 2.1. *Suppose that $f \in C^1(X,\mathbb{R})$ satisfies the (PS) condition and f has only finitely many critical points, then the following statements hold.*

(i) *If for some $k \in \mathbb{N}$ we have $C_k(f,\infty) \neq 0$, then f has a critical point u with $C_k(f,u) \neq 0$.*
(ii) *Let $\mathbf{0}$ be an isolated critical point of f. If for some $k \in \mathbb{N}$ we have $C_k(f,\mathbf{0}) \neq C_k(f,\infty)$, then f has a nonzero critical point.*

The rest of this section is devoted to the proof of Theorem 1.2. For this purpose, we recall the following Homotopy Invariant Theorem for critical groups at infinity.

Proposition 2.2 ([10, Theorem 3.1])**.** *Let $\{f_t : t \in [0,1]\}$ be a family of C^1-functionals defined on a Hilbert space X, which satisfies the (PS) condition, such that f_t' and $\partial_t f_t$ are locally Lipschitz continuous. If there exist $a \in \mathbb{R}$ and $\delta > 0$ such that*

$$f_t(u) \le a \implies \left\| f_t'(u) \right\| \ge \delta \quad \forall t \in [0,1], \tag{2.3}$$

then f_0^a is homeomorphic to f_1^a. In particular, for all $k \in \mathbb{N}$ we have $C_k(f_0,\infty) \cong C_k(f_1,\infty)$.

Now we are ready to prove Theorem 1.2. In what follows the letters C and $C_1, C_2, \cdots$ will be used to denote positive constants which are different from line to line.

2.1. *Proof of Theorem 1.2 (i)*

Let $A_\ell : X \to X$ be such that

$$\langle A_\ell(u), v\rangle = \int_\Omega (\nabla u \nabla v - \lambda_\ell uv)\,\mathrm{d}x, \qquad u, v \in X. \tag{2.4}$$

Then A_ℓ is a self-adjoint bounded linear operator on X. Let $X^0 = \ker(A_\ell)$, X^- and X^+ be the negative and positive spaces of A_ℓ respectively, then $X = X^+ \oplus X^0 \oplus X^-$.

For $u \in X$ we always write $u = u^+ + u^0 + u^-$ and $\widehat{u} = u^+ + u^0 - u^-$. It is well known that for $u^\pm \in X^\pm$,

$$\left.\begin{aligned} \int_\Omega \left(\left|\nabla u^+\right|^2 - \lambda_\ell (u^+)^2\right)\mathrm{d}x &\ge \mu \left\|u^+\right\|^2, \\ -\int_\Omega \left(|\nabla u^-|^2 - \lambda_\ell (u^-)^2\right)\mathrm{d}x &\ge (\lambda_\ell - \lambda_{\ell-1}) \int_\Omega (u^-)^2\,\mathrm{d}x, \end{aligned}\right\} \tag{2.5}$$

where μ is a positive number. Let

$$f_t(u) = (1-t)f(u) + t\left(\left\|u^+\right\|^2 + \left\|u^0\right\|^2 - \|u^-\|^2\right), \quad t \in [0,1].$$

We claim that for any $\{(t_n, u_n)\} \subset [0,1] \times X$,

$$\left.\begin{aligned} f_{t_n}'(u_n) &\to 0 \\ \|u_n\| &\to \infty \end{aligned}\right\} \implies f_{t_n}(u_n) \to +\infty. \tag{2.6}$$

To see this, set

$$q(x,t) = p(x,t) - \lambda_\ell t, \qquad Q(x,t) = \int_0^t q(x,s)\mathrm{d}s.$$

Then for $|u| \ge R$ we have

$$\lambda_{\ell-1} + \varepsilon - \lambda_\ell \le \frac{q(x,u)}{u} \le 0.$$

It follows that for $|u| \ge R$

$$q(x,u)\widehat{u} = \frac{q(x,u)}{u} u\widehat{u} = \frac{q(x,u)}{u}\left((u^+ + u^0)^2 - (u^-)^2\right)$$

$$\leq (\lambda_\ell - \lambda_{\ell-1} - \varepsilon)(u^-)^2.$$

Hence

$$\begin{aligned} -\int_\Omega q(x,u)\widehat{u}\mathrm{d}x &= -\int_{|u|\geq R} q(x,u)\widehat{u}\mathrm{d}x - \int_{|u|\leq R} q(x,u)\widehat{u}\mathrm{d}x \\ &\geq -(\lambda_\ell - \lambda_{\ell-1} - \varepsilon)\int_{|u|\geq R} (u^-)^2\,\mathrm{d}x - C\,\|\widehat{u}\| \\ &\geq -(\lambda_\ell - \lambda_{\ell-1} - \varepsilon)\int_\Omega (u^-)^2\,\mathrm{d}x - C\,\|\widehat{u}\| \\ &= -(\lambda_\ell - \lambda_{\ell-1} - \varepsilon)\int_\Omega (u^-)^2\,\mathrm{d}x - C\,\|u\|. \end{aligned}$$

Thus according to (2.5), since $\varepsilon \in (0, \lambda_\ell - \lambda_{\ell-1})$, we have

$$\begin{aligned} \|u_n\| &\geq \left\langle f'_{t_n}(u_n), \widehat{u}_n \right\rangle \\ &= (1-t_n)\left[\int_\Omega \left(\nabla u_n \nabla \widehat{u}_n - \lambda_\ell u_n \widehat{u}_n\right)\mathrm{d}x - \int_\Omega q(x,u_n)\widehat{u}_n\,\mathrm{d}x\right] + t_n\,\|u_n\|^2 \\ &\geq (1-t_n)\left[\mu\,\|u^+\|^2 + \varepsilon\int_\Omega (u_n^-)^2\,\mathrm{d}x - C\,\|u_n\|\right] + t_n\,\|u_n\|^2 \\ &\geq (1-t_n)C_1\left(\|u_n^+\|^2 + \|u_n^-\|^2\right) - C_2\,\|u_n\| + t_n\,\|u_n\|^2. \end{aligned} \tag{2.7}$$

It follows that $t_n \to 0$, $\|u_n^+\|^2 + \|u_n^-\|^2 \leq C\,\|u_n^0\|$. Hence

$$\begin{aligned} \frac{1}{\|u_n\|}\int_\Omega \left(|\nabla u_n|^2 - \lambda_\ell u^2\right)\mathrm{d}x &\geq \frac{1}{\|u_n\|}\int_\Omega \left(|\nabla u_n^-|^2 - \lambda_\ell (u_n^-)^2\right)\mathrm{d}x \\ &\geq \frac{-C_1\,\|u_n^-\|^2}{\|u_n\|} \geq -C. \end{aligned} \tag{2.8}$$

Let $z_n = u_n/\,\|u_n\|$, then $\|z_n\| = 1$. Hence up to a subsequence

$$z_n \rightharpoonup z \text{ in } X, \qquad z_n(x) \to z(x) \text{ a.e. in } \Omega.$$

We write

$$z_n = \frac{u_n^+}{\|u_n\|} + \frac{u_n^-}{\|u_n\|} + \frac{u_n^0}{\|u_n\|}.$$

Obviously, (2.7) also implies

$$\left\|\frac{u_n^+}{\|u_n\|}\right\| \to 0, \qquad \left\|\frac{u_n^-}{\|u_n\|}\right\| \to 0, \qquad \left\|\frac{u_n^0}{\|u_n\|}\right\| \to 1. \tag{2.9}$$

Since $\dim X^0 < \infty$, we see that $\left\{\frac{u_n^0}{\|u_n\|}\right\}$ has a subsequence which converges to a point in X^0. Hence up to a subsequence $z_n \to z$ in X with $\|z\| = 1$. Thus $z \neq \mathbf{0}$.

For $x \in \Theta := \{x \in \Omega :\ z(x) \neq 0\}$, we have $|u_n(x)| \to +\infty$. Thus we have

$$\frac{Q(x,u_n(x))}{|u_n(x)|}\frac{|u_n(x)|}{\|u_n\|} = \frac{Q(x,u_n(x))}{|u_n(x)|}(|z(x)| + o(1)) \to -\infty.$$

By the Fatou's Lemma, since the Lebesgue measure of Θ is positive, we have

$$\frac{1}{\|u_n\|}\int_{z(x)\neq 0} Q(x,u_n(x))\,\mathrm{d}x = \int_{z(x)\neq 0} \frac{Q(x,u_n(x))}{|u_n(x)|}\frac{|u_n(x)|}{\|u_n\|}\,\mathrm{d}x \to -\infty. \tag{2.10}$$

On the other hand, $Q(x,t)$ is bounded from above, hence for n large enough,

$$\frac{1}{\|u_n\|}\int_{z(x)=0} Q(x,u_n(x))\,\mathrm{d}x \le C. \tag{2.11}$$

From (2.8), (2.10) and (2.11) we have

$$\begin{aligned} f(u_n) &= \|u_n\|\left(\frac{1}{\|u_n\|}\int_\Omega \left(|\nabla u_n|^2-\lambda_\ell u^2\right)\mathrm{d}x - \frac{1}{\|u_n\|}\int_\Omega Q(x,u_n(x))\,\mathrm{d}x\right) \\ &= \|u_n\|\left(\frac{1}{\|u_n\|}\int_\Omega \left(|\nabla u_n|^2-\lambda_\ell u^2\right)\mathrm{d}x\right. \\ &\qquad \left. -\frac{1}{\|u_n\|}\left(\int_{z(x)\neq 0}+\int_{z(x)=0}\right)Q(x,u_n(x))\,\mathrm{d}x\right) \to +\infty. \end{aligned} \tag{2.12}$$

Finally, (2.9) implies that for n large enough,

$$\left\|u_n^+\right\|^2+\left\|u_n^0\right\|^2-\|u_n^-\|^2>0.$$

Note that $t_n\to 0$, we have

$$\begin{aligned} f_{t_n}(u_n) &= (1-t_n)f(u_n)+t_n\left(\left\|u_n^+\right\|^2+\left\|u_n^0\right\|^2-\|u_n^-\|^2\right) \\ &\ge \frac{1}{2}f(u_n)\to+\infty. \end{aligned}$$

So we have proved our claim (2.6). This implies that the family of C^1-functionals $\{f_t : t\in[0,1]\}$ satisfies the (PS) condition and (2.3). Thus $f=f_0$ also satisfies the (PS) condition and, by Proposition 2.2,

$$C_k(f,\infty)=C_k(f_0,\infty)\cong C_k(f_1,\infty)\cong\delta_{k,\dim X^-}\mathscr{G}=\delta_{k,\ell-1}\mathscr{G}.$$

This completes the proof of Theorem 1.2 (i).

2.2. *Proof of Theorem 1.2 (ii)*

Decompose the space X as before, $X=X^+\oplus X^0\oplus X^-$. Let

$$h(u)=\frac{1}{2}\int_\Omega\left(|\nabla u|^2-\lambda_\ell u^2\right)\mathrm{d}x-\left\|u^0\right\|^2,$$

and define a family of C^{2-0}-functionals $\{f_t : t\in[0,1]\}$ on X:

$$\begin{aligned} f_t(u) &= (1-t)f(u)+th(u) \\ &= \frac{1}{2}\int_\Omega\left(|\nabla u|^2-\lambda_\ell u^2\right)\mathrm{d}x-(1-t)\int_\Omega Q(x,u)\,\mathrm{d}x-t\left\|u^0\right\|^2. \end{aligned} \tag{2.13}$$

Then arguing as in §2.1 (now for $u=u^++u^0+u^-$, we set $\widehat{u}=u^+-u^0-u^-$), we can prove a claim ‘dual’ to (2.6):

$$\left.\begin{array}{r} f'_{t_n}(u_n)\to 0 \\ \|u_n\|\to\infty \end{array}\right\} \Longrightarrow f_{t_n}(u_n)\to-\infty. \tag{2.14}$$

Remark 2.3. The claim (2.14) implies that f satisfies the (PS) condition, and the critical values of f are bounded from above.

Although we have the claim (2.14), we can not say anything about the relation between $C_*(f_0,\infty)$ and $C_*(f_1,\infty)$. This is why Li-Perera-Su could not compute $C_*(f,\infty)$ explicitly in [10, Proposition 1.3].

Thanks to [14, Theorem 1.1], the condition (1.2) allows us to reduce the problem into a finite dimensional subspace. Then we can use the Alexander Duality Theorem to overcome this difficulty.

To carry out this program, let us consider a C^1-functional $\varphi : X \to \mathbb{R}$, which satisfies the (PS) condition. If the critical values of φ are bounded from below by some $\alpha \in \mathbb{R}$, then we can define a new sequence of critical groups using singular cohomology:

$$\mathfrak{C}^k(\varphi,\infty) = H^k(X,\varphi^\alpha), \qquad k \in \mathbb{N}.$$

If the critical values of φ are bounded, then both $C_*(\varphi,\infty)$ and $\mathfrak{C}^*(-\varphi,\infty)$ make sense. In the case $X = \mathbb{R}^m$, they have the following dual relation:

Lemma 2.4. *If $\varphi \in C^1(\mathbb{R}^m,\mathbb{R})$ satisfies the (PS) condition, with critical values bounded, then $C_k(\varphi,\infty) \cong \mathfrak{C}^{m-k}(-\varphi,\infty)$.*

Proof. Suppose that the critical values of φ are contained in the interval $(-\alpha,\alpha)$, where α is a positive number. Since φ and $-\varphi$ have no critical value greater than α, both φ^α and $(-\varphi)^\alpha$ are strong deformation retractor of $\mathbb{R}^m$, hence

$$\begin{aligned} C_k(\varphi,\infty) &= H_k(\mathbb{R}^m,\varphi^{-\alpha}) \cong H_k(\{\varphi<\alpha\},\{\varphi<-\alpha\}),\\ \mathfrak{C}^k(-\varphi,\infty) &= H^k\big(\mathbb{R}^m,(-\varphi)^{-\alpha}\big) \cong H^k\big((-\varphi)^\alpha,(-\varphi)^{-\alpha}\big). \end{aligned}$$

Let $\mathfrak{p} : \mathbb{R}^m \to S^m$ be the stereographic projection from the north pole $p \in S^m$, applying the Alexander Duality Theorem, we have

$$\begin{aligned} C_k(\varphi,\infty) &\cong H_k(\{\varphi<\alpha\},\{\varphi<-\alpha\})\\ &\cong H_k\big(\mathfrak{p}(\{\varphi<\alpha\}),\mathfrak{p}(\{\varphi<-\alpha\})\big)\\ &\cong \check{H}^{m-k}\big(S^m\backslash\mathfrak{p}(\{\varphi<-\alpha\}),S^m\backslash\mathfrak{p}(\{\varphi<\alpha\})\big), \end{aligned}$$

where $\check{H}^*$ stands for the Čech cohomology. Because respect to the Čech cohomology, any closed subset is excisable from a locally compact pair in an ENR (euclidean neighborhood retract), we obtain

$$\begin{aligned} C_k(\varphi,\infty) &\cong \check{H}^{m-k}\big(S^m\backslash\mathfrak{p}(\{\varphi<-\alpha\})\backslash\{p\},S^m\backslash\mathfrak{p}(\{\varphi<\alpha\})\backslash\{p\}\big)\\ &= \check{H}^{m-k}\big(\mathfrak{p}(\mathbb{R}^m\backslash\{\varphi<-\alpha\}),\mathfrak{p}(\mathbb{R}^m\backslash\{\varphi<\alpha\})\big)\\ &\cong \check{H}^{m-k}\big(\mathbb{R}^m\backslash\{\varphi<-\alpha\},\mathbb{R}^m\backslash\{\varphi<\alpha\}\big)\\ &= \check{H}^{m-k}\big((-\varphi)^\alpha,(-\varphi)^{-\alpha}\big)\\ &\cong H^{m-k}\big((-\varphi)^\alpha,(-\varphi)^{-\alpha}\big) \cong \mathfrak{C}^{m-k}(-\varphi,\infty). \end{aligned} \tag{2.15}$$

In step (2.15) we use the fact that $\big((-\varphi)^\alpha,(-\varphi)^{-\alpha}\big)$ is a pair of ENR's, hence according to [7, Proposition 8.6.12], the Čech cohomology coincides with the singular cohomology. ☐

Remark 2.5. The above lemma is motivated by [5], where similar result was obtained for the critical groups $C_*(f,u)$ at an isolated critical point.

To reduce the problem to a finite dimensional space, we recall the following proposition, which is based on the Lyapunov-Schmidt reduction [1, 3].

Proposition 2.6 ([14, Theorem 1.1]). *Let X be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, Y, Z be closed subspace of X such that $X = Z \oplus Y$. Assume that $f \in C^1(X,\mathbb{R})$ satisfies the (PS) condition, the critical values of f are bounded from below. If there is a real number $\kappa > 0$ such that for all $z \in Z$ and $y_1, y_2 \in Y$, there holds*

$$\langle \nabla f(z+y_1) - \nabla f(z+y_2), y_1 - y_2\rangle \geq \kappa \|y_1 - y_2\|^2,$$

then there exists a C^1-functional $\varphi : Z \to \mathbb{R}$ which satisfies (PS) and with critical values bounded from below, such that $C_k(f,\infty) \cong C_k(\varphi,\infty)$ for all $k \in \mathbb{N}$.

In fact, $\varphi : Z \to \mathbb{R}$ is given by

$$\varphi(z) = f(z+\psi(z)) = \min_{y\in Y} f(z+y),$$

where $\psi \in C(Z,Y)$. For the given $z \in Z$, $\psi(z)$ is the unique element in Y satisfying the above equation. Let $P_Z : X \to Z$ and $P_Y : X \to Y$ denote the orthogonal projection, we also know that for all $z \in Z$, there holds

$$P_Z \nabla f(z+\psi(z)) = \nabla\varphi(z), \qquad P_Y \nabla f(z+\psi(z)) = 0. \tag{2.16}$$

See [1, 3] for the detail.

Remark 2.7. According to [1, Corollary 2.4 (ii)], if $f \in C^{2-0}_{\mathrm{loc}}(X,\mathbb{R})$, then $\psi \in C^{1-0}_{\mathrm{loc}}(Z,Y)$. Hence $\varphi \in C^{2-0}_{\mathrm{loc}}(Z,\mathbb{R})$.

Proof of Theorem 1.2 (ii). By (1.2), for $x \in \Omega$, $t_1, t_2 \in \mathbb{R}$ we have

$$\frac{q(x,t_1) - q(x,t_2)}{t_1 - t_2} \leq \Lambda - \lambda_\ell, \tag{2.17}$$

where $q(x,t) = p(x,t) - \lambda_\ell t$. Since $\lim_{i\to\infty} \lambda_i = +\infty$, we can choose some $r > \ell$, such that

$$\max_{t\in[0,1]} \big(\lambda_\ell + (1-t)(\Lambda - \lambda_\ell)\big) = \max(\Lambda, \lambda_\ell) \leq \lambda_r < \lambda_{r+1}. \tag{2.18}$$

Decompose the space X as $X = Z \oplus Y$, where Y is the positive space of A_r and Z is the orthogonal complement of Y. Note that here A_r is defined similar to (2.4). Consider the family of functionals $\{f_t : t \in [0,1]\}$ introduced in (2.13). Since

$$f_t(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda_\ell u^2\right) \mathrm{d}x - (1-t)\int_\Omega Q(x,u)\,\mathrm{d}x - t\left\|u^0\right\|^2$$

and $u^0 \in X^0 \subset Z$, that is, $u^0 \perp Y$, using (2.17), we see that for $z \in Z$, $y_1, y_2 \in Y$ there holds

$$\begin{aligned}
&\langle \nabla f_t(z+y_1) - \nabla f_t(z+y_2), y_1 - y_2\rangle \\
&\quad = \int_\Omega |\nabla(y_1-y_2)|^2\,\mathrm{d}x - \lambda_\ell \int_\Omega (y_1-y_2)^2\,\mathrm{d}x \\
&\qquad\qquad - (1-t)\int_\Omega \big(q(x,z+y_1) - q(x,z+y_2)\big)(y_1-y_2)\,\mathrm{d}x \\
&\quad \geq \int_\Omega |\nabla(y_1-y_2)|^2\,\mathrm{d}x - \lambda_\ell \int_\Omega (y_1-y_2)^2\,\mathrm{d}x \\
&\qquad\qquad - (1-t)(\Lambda-\lambda_\ell)\int_\Omega (y_1-y_2)^2\,\mathrm{d}x \\
&\quad = \int_\Omega |\nabla(y_1-y_2)|^2\,\mathrm{d}x - \big(\lambda_\ell + (1-t)(\Lambda-\lambda_\ell)\big)\int_\Omega (y_1-y_2)^2\,\mathrm{d}x \geq \kappa\|y_1-y_2\|^2,
\end{aligned} \tag{2.19}$$

where

$$\kappa = 1 - \frac{\max(\Lambda, \lambda_\ell)}{\lambda_{r+1}} > 0.$$

Applying Proposition 2.6, we obtain a family of C^{2-0}_{loc}-functionals $\{\varphi_t : t \in [0,1]\}$ defined on Z. The critical values of each φ_t are bounded from below, and

$$C_k(f_t, \infty) \cong C_k(\varphi_t, \infty), \qquad \forall k \in \mathbb{N}. \tag{2.20}$$

To see that $\partial_t \varphi_t \in C^{1-0}_{\mathrm{loc}}(Z, \mathbb{R})$, we write $F(t,u) = f_t(u)$. Then $F : \mathbb{R} \times X \to \mathbb{R}$ is of class C^{2-0}, and

$$\nabla F(t,u) = P_{\mathbb{R}} \nabla F(t,u) + P_X \nabla F(t,u) = P_{\mathbb{R}} \nabla F(t,u) + \nabla f_t(u). \tag{2.21}$$

According to the decomposition $\mathbb{R} \times X = (\mathbb{R} \oplus Z) \oplus Y$, since $P_{\mathbb{R}} \nabla F(t,u) \perp Y$, by (2.19) and (2.21) we have

$$\begin{aligned} &\langle \nabla F(t+z+y_1) - \nabla F(t+z+y_2), y_1 - y_2 \rangle \\ &\quad = \langle \nabla f_t(z+y_1) - \nabla f_t(z+y_2), y_1 - y_2 \rangle \ge \kappa \left\| y_1 - y_2 \right\|^2. \end{aligned}$$

Hence there exists $\Phi \in C^{2-0}_{\mathrm{loc}}(\mathbb{R} \times Z, \mathbb{R})$, such that

$$\Phi(t,z) = \max_{y \in Y} F(t,z,y) = \max_{y \in Y} f_t(z+y) = \varphi_t(z).$$

So it is clear that $\partial_t \varphi_t = \partial_t \Phi(t,\cdot) \in C^{1-0}_{\mathrm{loc}}(Z, \mathbb{R})$.

We claim that

$$\left. \begin{array}{r} \nabla \varphi_{t_n}(z_n) \to 0 \\ \|z_n\| \to \infty \end{array} \right\} \Longrightarrow (-\varphi_{t_n})(z_n) \to +\infty. \tag{2.22}$$

In fact, since $\nabla \varphi_{t_n}(z_n) \to 0$, using (2.16) we have

$$\begin{aligned} \nabla f_{t_n}(z_n + \psi(z_n)) &= P_Z \nabla f_{t_n}(z_n + \psi(z_n)) + P_Y \nabla f_{t_n}(z_n + \psi(z_n)) \\ &= \nabla \varphi_{t_n}(z_n) \to 0. \end{aligned}$$

Moreover,

$$\|z_n + \psi(z_n)\| \ge \|z_n\| \to \infty.$$

Hence by our claim (2.14) we have

$$(-\varphi_{t_n})(z_n) = (-f_{t_n})(z_n + \psi(z_n)) \to +\infty.$$

This proves our claim (2.22). So the critical values of each φ_t are bounded.

By Proposition 2.2, $(-\varphi_0)^a$ is homeomorphic to $(-\varphi_1)^a$. Hence

$$\mathfrak{C}^k(-\varphi_0, \infty) \cong \mathfrak{C}^k(-\varphi_1, \infty), \qquad \forall k \in \mathbb{N}.$$

Since $m = \dim Z < \infty$, applying Lemma 2.4 we obtain

$$\begin{aligned} C_k(\varphi_0, \infty) &\cong \mathfrak{C}^{m-k}(-\varphi_0, \infty) \\ &\cong \mathfrak{C}^{m-k}(-\varphi_1, \infty) \cong C_k(\varphi_1, \infty), \qquad \forall k \in \mathbb{N}. \end{aligned}$$

Finally, by (2.20) we see that

$$\begin{aligned} C_k(f, \infty) &= C_k(f_0, \infty) \\ &\cong C_k(\varphi_0, \infty) \cong C_k(\varphi_1, \infty) \\ &\cong C_k(f_1, \infty) = C_k(h, \infty) \cong \delta_{k,\ell} \mathscr{G}, \qquad \forall k \in \mathbb{N}. \end{aligned}$$

This completes the proof of Theorem 1.2 (ii). □

3. Solutions for elliptic resonant problems

Having computed the critical groups of f at infinity, we apply the Morse theory to give the proof of Theorem 1.1.

Proof of Theorem 1.1. We only prove the theorem for Case (ii). For this purpose it suffices to find a nonzero critical point for the functional $f : H_0^1(\Omega) \to \mathbb{R}$,

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega P(x,u)\,\mathrm{d}x.$$

Under our assumptions it is well known that $f \in C^{2-0}(H_0^1(\Omega),\mathbb{R})$.

Let Y be the positive space of A_m and Z the orthogonal complement of Y. According to [13, Lemma 4.2], the condition (p_0) implies that f has a local linking at $\mathbf{0}$ with respect to the decomposition $X = Z \oplus Y$. That is, there is $\delta > 0$ such that for $\|u\| \le \delta$,

$$f(u) \le 0, \quad \text{if } u \in Z; \qquad f(u) > 0, \quad \text{if } u \in Y \setminus \{\mathbf{0}\}.$$

By [12, Theorem 2.1], since $\dim Z = m$ we deduce

$$C_m(f,\mathbf{0}) \neq 0.$$

On the other hand, by Theorem 1.2 (ii) we know that f satisfies the (PS) condition and

$$C_k(f,\infty) \cong \delta_{k,\ell}\mathcal{G}.$$

Bur $m \neq \ell$, we have

$$C_m(f,\mathbf{0}) \neq C_m(f,\infty).$$

Using Proposition 2.1 (ii), we see that f has a nonzero critical point u. □

The existence of multiple solutions requires more conditions on the nonlinearity. In what follows we will assume more regularity conditions on p, so that the functional f is of class C^2.

Theorem 3.1. *Suppose $p \in C^1(\Omega \times \mathbb{R},\mathbb{R})$ and there exists $\Lambda > 0$ such that $|p'_t(x,t)| \le \Lambda$. If $m > 1$ and there exists $\rho > 0$ such that*

$$\lambda_m \le \frac{p(x,t)}{t} \le \lambda_{m+1} \qquad \text{for } 0 < |t| \le \rho, \tag{3.1}$$

then the problem (1.1) has at least two nontrivial solutions in one of the following cases:

(i) *p satisfies (p_-) with $\ell = 2$.*
(ii) *p satisfies (p_+) with $\ell = 1$.*

Proof. We only prove the theorem for Case (i). Under the assumptions we see that f is of class C^2. According to [10, Proposition 1.1], the condition (3.1) implies that

$$C_k(f,\mathbf{0}) \cong \delta_{k,m}\mathcal{G}. \tag{3.2}$$

On the other hand, by Theorem 1.2 (i), f satisfies (PS) and

$$C_k(f,\infty) \cong \delta_{k,1}\mathcal{G}.$$

In particular $C_1(f,\infty) \neq 0$. By Proposition 2.1 (i) we see that f has a critical point u with $C_1(f,u) \neq 0$. That is, u is a mountain pass type critical point of f. Since f is of class C^2, a standard argument as in [4, 6] shows that

$$C_k(f,u) \cong \delta_{k,1}\mathcal{G}.$$

Comparing this with (3.2) we see that $u \neq \mathbf{0}$.

If $\mathbf{0}$ and u are the only critical points of f, then the Morse inequality (2.2) with $t = -1$ gives

$$(-1)^m + (-1)^1 = (-1)^1 .$$

This is impossible. Therefore f has at least one more critical point. □

Remark 3.2. The reason that we have to restrict on the case $\ell \le 2$ is that, for $h > 1$ we don't know whether $C_h(f, u) \neq 0$ implies $C_k(f, u) \cong \delta_{k,h}\mathscr{G}$, so we can not compute $C_k(f, u)$ explicitly.

Assuming the limit (1.5), with different assumptions on the nonlinearity similar results have been obtained by Robinson [17] via topological degree and by Su-Zhao [19] using Morse theory. Our Theorem 3.1 does not require that the above limit exists, therefore it can be applied to more general case. For example, choose $\lambda \in \left(0, \frac{\lambda_1+\lambda_2}{2}\right)$. Let $p \in C^1(\mathbb{R}, \mathbb{R})$ be such that $p'(0) > \lambda_2$ is not an eigenvalue and

$$p(t) = \lambda t + (\lambda - \lambda_1)t \sin \ln |t|$$

for $|t|$ large, then the limit (1.5) does not exist. But it is easy to verify that Theorem 3.1 (ii) can be applied and we still obtian two nontrivial solutions.

References

[1] H. Amann, Saddle points and multiple solutions of differential equations, *Math. Z.* 169 (1979), 127–166.

[2] T. Bartsch, S. J. Li, Critical point theory for asymptotically quadratic functionals and applications to problems with resonance, *Nonl. Anal., TMA*, 28 (1997), 419–441.

[3] A. Castro, Reduction methods via minimax, *Differential equations*, pp. 1–20, Lecture Notes in Math., 957, Springer, Berlin-New York, 1982.

[4] K. C. Chang, Infinite dimensional Morse theory and multiple solution problem, Birkhäuser, Bostom, 1993.

[5] K. C. Chang, Critical groups, Morse theory and application to semilinear elliptic BVPs, *Chinese mathematics into the 21st century*, 41–65, Peking Univ. Press, Beijing, 1991.

[6] K. C. Chang, S. J. Li and J. Q. Liu, Remarks on multiple solutions for asymptotically linear elliptic boundary value problems, *Topological Methods in Nonl. Anal.*, 3 (1994), 179–187.

[7] A. Dold, Lectures on algebraic topology, (2nd edition), Springer, Berlin, 1980.

[8] E. Landesman, A. Lazer, Nonlinear perturbations of linear eigenvalues problem at resonance, *J. Math. Mech.*, 19 (1970) 609–623.

[9] S. J. Li, J. Q. Liu, Computations of critical groups at degenerate critical point and applications to nonlinear differential equations with resonance, *Houston J. Math.*, 25 (1999), 563–582.

[10] S. J. Li, K. Perera, J. B. Su, Computation of critical groups in elliptic boundary-value problems where the asymptotic limits may not exist, *Proc. Roy. Soc. Edinburgh*, 131A (2001), 721–732.

[11] S. J. Li, W. M. Zou, The computations of the critical groups with an application to elliptic resonant problems at a higher eigenvalue, *J. Math. Anal. Appl.*, 235 (1999), 237–259.

[12] J. Q. Liu, A Morse index for a saddle point, *Syst. Sc. Math. Sc.*, 2 (1989), 32–39.

[13] J. Q. Liu, J. B. Su, Remarks on multiple nontrivial solutions for quasilinear resonant problems, *J. Math. Anal. Appl.*, 258 (2001), 209–222.

[14] S. B. Liu, S. J. Li, Critical groups at infinity, saddle point reduction and elliptic resonant problems, *Commun. Contemp. Math.*, 5 (2003), 761–773.

[15] J. Mawhin, M. Willem, Critical point theory and Hamiltonian systems, Springer, Berlin, 1989.

[16] K. Perera, M. Schechter, Solution of nonlinear equations having asymptotic limits at zero and infinity, *Calc. Var.*, 12 (2001), 359–369.

[17] S. Robinson, Multiple solutions for semilinear elliptic boundary value problems at resonance, *Electron. J. Differential Equations*, 1995 (01) (1995) 1–14.

[18] J. B. Su, Semilinear elliptic resonant problems at higher eigenvalue with unbounded nonlinear terms, *Acta Math. Sinica, (New Ser.)*, 14 (1998), 411–418.

[19] J. B. Su, L. G. Zhao, An elliptic resonance problem with multiple solutions, *J. Math. Anal. Appl.*, 319 (2006), 604–616.