

CRITICAL GROUPS AT INFINITY, SADDLE POINT REDUCTION AND ELLIPTIC RESONANT PROBLEMS

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We prove that if the Lyapunov–Schmidt method is applicable, then the critical groups at infinity of the reduced functional are isomorphic to those of the original functional. This observation, combining with our results on the critical groups of finite dimensional functional, enables us to compute the critical groups at infinity. New multiplicity results for resonant elliptic boundary value problems are obtained as applications.

Keywords: Morse inequality; critical group; Lyapunov–Schmidt reduction; elliptic resonant problem.

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1. Introduction

Infinite dimensional Morse theory is a very useful tool in treating multiple solution problems in the study of nonlinear differential equations. For a systematic exploration of this theory, the reader is referred to the books by Chang [5] and Mawhin–Willem [13]. The main concept in this theory is the *critical group* $C_q(f, u)$ for a C^1 -functional $f : X \rightarrow \mathbf{R}$ at an isolated critical point u , where X is a Banach space. In [3], Bartsch and Li introduced the *critical group at infinity*, $C_q(f, \infty)$. Using these concepts, we have the following famous *Morse inequality*

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1+t)Q(t), \quad (1.1)$$

where Q is a formal series with nonnegative coefficients,

$$M_q = \sum_{u \in K} \text{rank } C_q(f, u), \quad \beta_q = \text{rank } C_q(f, \infty),$$

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being $K := \{u \in X : f'(u) = 0\}$, the critical set of f . In most applications, unknown critical points will be found from (1.1) if we can compute both the critical groups at known critical points and the critical groups at infinity. Thus the computation of the critical groups is very important.

On the other hand, the Lyapunov–Schmidt reduction procedure is a powerful method on finding critical points for a C^1 -functional $f : X \rightarrow \mathbf{R}$, where $X = X^- \oplus X^+$ is a separable Hilbert space being X^- , X^+ closed subspaces of X . In most applications, X^- or X^+ is finite dimensional. Under appreciative conditions, there exists a reduced functional $\varphi : X^- \rightarrow \mathbf{R}$, which is closely related to the study of the original functional f . The purpose of this paper is to reveal this relation further from the Morse theoretical point of view. More precisely, we have the following abstract results:

Theorem 1.1. *Let X be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$, X^- , X^+ be closed subspaces of X such that $X = X^- \oplus X^+$. Assume that $f \in C^1(X, \mathbf{R})$ satisfies the (PS) condition, the critical values of f are bounded from below. If there is a real number $m > 0$ such that for all $v \in X^-$ and $w_1, w_2 \in X^+$, there holds*

$$\langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq m\|w_1 - w_2\|^2, \tag{1.2}$$

then there exists a C^1 -functional $\varphi : X^- \rightarrow \mathbf{R}$ such that

$$C_q(f, \infty) \cong C_q(\varphi, \infty), \quad q = 0, 1, 2, \dots$$

Moreover, if $k := \dim X^- < \infty$ and $C_k(f, \infty) \neq 0$, then $C_q(f, \infty) \cong \delta_{q,k}\mathcal{G}$.

Theorem 1.2. *Let X be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$, X^- , X^+ be closed subspaces of X such that $X = X^- \oplus X^+$ and $j := \dim X^+ < \infty$. Assume that $f \in C^1(X, \mathbf{R})$ satisfies the (PS) condition, the critical values of f are bounded from below. If there is a real number $m > 0$ such that for all $v \in X^-$ and $w_1, w_2 \in X^+$, there holds*

$$-\langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq m\|w_1 - w_2\|^2, \tag{1.3}$$

then there exists a C^1 -functional $\varphi : X^- \rightarrow \mathbf{R}$ such that

$$C_q(f, \infty) \cong C_{q-j}(\varphi, \infty), \quad q = 0, 1, 2, \dots$$

Moreover, if $C_j(f, \infty) \neq 0$, then $C_q(f, \infty) \cong \delta_{q,j}\mathcal{G}$.

We will recall some notions and results on Morse theory and the Lyapunov–Schmidt reduction method in Sec. 2. The proofs of Theorems 1.1 and 1.2 will also be presented in this section. We remark that, there are some recent works concerning on the computation of the critical groups $C_q(f, \infty)$, cf. [3, 5, 9, 10]. Theorems 1.1 and 1.2 enable us to give new computations of these critical groups.

In Sec. 3, we apply our abstract results to the following semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = p(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded open domain with smooth boundary $\partial\Omega$ and $p : \mathbf{R} \rightarrow \mathbf{R}$ is a C^1 -function. We are interested in the case that the problem (1.4) is asymptotically linear at both zero and infinity.

Since the pioneer work of Amann and Zehnder [1], where the existence of at least one nontrivial solution was proved if the nonlinearity crosses at least one eigenvalue, this kind of problems have captured great interest. The resonance case, i.e., $\lim_{|u| \rightarrow \infty} p'(u) = \lambda_\ell$ for some $\ell \in \mathbf{N}$, is more delicate. In this case, the existence of at least one nontrivial solution has been obtained in [3, 9–11, 15, 16].

As applications of our abstract results, in the case that $p'(0) < \lambda_1$, we obtain at least four nontrivial solutions of problem (1.4). Such results have been obtained in [12], provided that the nonlinearity $p(u) - \lambda_\ell u$ is bounded or grows slower than $|u|^\alpha$, for some $\alpha \in (0, 1)$. Our result do not have this restriction.

2. Critical Groups, Saddle Point Reduction

Let X be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$, $f \in C^1(X, \mathbf{R})$. Let u_0 be an isolated critical point of f with $f(u_0) = c$. Then the group

$$C_q(f, u_0) := H_q(f_c, f_c \setminus \{u_0\}), \quad q = 0, 1, 2, \dots$$

is called the q th *critical group* of f at u_0 , here $f_c = \{u \in X : f(u) \leq c\}$, $H_q(A, B)$ stands for the q th singular relative homology group of the topological pair (A, B) with coefficients in an Abelian group \mathcal{G} .

If $f \in C^2(X, \mathbf{R})$, then $f''(u_0)$ is a self-adjoint linear operator on X . The dimension of the largest negative space of $f''(u_0)$, is called the *Morse index* of f at u_0 , the dimension of the kernel of $f''(u_0)$ is called the *nullity* of f at u_0 . We say that u_0 is *nondegenerate* if and only if the nullity of f at u_0 is zero. For a nondegenerate critical point u_0 with Morse index μ , we have $C_q(f, u_0) \cong \delta_{q,\mu} \mathcal{G}$. For a critical point, which may be degenerate, we have the following important result:

Proposition 2.1 ([13, Corollary 8.4]). *Suppose $f \in C^2(X, \mathbf{R})$, u_0 is an isolated critical point with finite Morse index μ and nullity ν . If $f''(u_0)$ is a Fredholm operator, then $C_q(f, u_0) \cong 0$, for $q \notin [\mu, \mu + \nu]$. Moreover,*

- (1) $C_\mu(f, u) \neq 0$ implies $C_q(f, u) \cong \delta_{q,\mu} \mathcal{G}$,
- (2) $C_{\mu+\nu}(f, u) \neq 0$ implies $C_q(f, u) \cong \delta_{q,\mu+\nu} \mathcal{G}$.

Let us recall a compact condition in the sense of Cerami, which is weaker than the usual compact condition (*PS*).

Definition 2.1. We say that $f \in C^1(X, \mathbf{R})$ satisfies the condition (C), if, any sequence $\{u_n\} \subset X$ such that $\{f(u_n)\}$ is bounded and $(1 + \|u_n\|)\|f'(u_n)\| \rightarrow 0$ possesses a convergent subsequence.

Now we assume that the critical values of f are strictly bounded from below by some $\alpha \in \mathbf{R}$, and f satisfies the condition (C). Then the q th critical group at infinity of f is defined in [3, Definition 3.4] as

$$C_q(f, \infty) := H_q(X, f_\alpha), \quad q = 0, 1, 2, \dots$$

Due to the condition (C), these groups are not dependent on the choice of α .

Let us recall a global version of the Lyapunov–Schmidt method. For the proof, the reader is referred to [4, Lemma 2.1], see also [2].

Proposition 2.2. Let X be a separable Hilbert space, X^-, X^+ be closed subspaces of X such that $X = X^- \oplus X^+$. Let $f : X \rightarrow \mathbf{R}$ be a C^1 -functional. If there is a real number $m > 0$ such that for all $v \in X^-$ and $w_1, w_2 \in X^+$, there holds

$$\pm \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq m \|w_1 - w_2\|^2, \tag{E_\pm}$$

then we have

(i) There exists a continuous function $\psi : X^- \rightarrow X^+$ such that

$$\text{if } (E_+) \text{ holds, then } f(v + \psi(v)) = \min_{w \in X^+} f(v + w),$$

$$\text{if } (E_-) \text{ holds, then } f(v + \psi(v)) = \max_{w \in X^+} f(v + w).$$

Moreover, $\psi(v)$ is the unique member of X^+ such that

$$\langle \nabla f(v + \psi(v)), w \rangle = 0, \quad \text{for all } w \in X^+.$$

(ii) The functional $\varphi : X^- \rightarrow \mathbf{R}$ defined by $\varphi(v) = f(v + \psi(v))$ is of class C^1 , and

$$\langle \nabla \varphi(v), v_1 \rangle = \langle \nabla f(v + \psi(v)), v_1 \rangle, \quad \text{for all } v, v_1 \in X^-.$$

(iii) An element $v \in X^-$ is a critical point of φ if and only if $v + \psi(v)$ is a critical point of f .

Remark 2.1. From Proposition 2.2, it is easy to see that, if the critical values of f are bounded from below, then the critical values of φ are also bounded from below; if f satisfies the (PS) condition, so does φ , cf. [2, Lemma 1].

Now we are in a position to state the proofs of Theorems 1.1 and 1.2. The following lemmas will be needed.

Lemma 2.1. Assume that $\varphi \in C^1(\mathbf{R}^n, \mathbf{R})$ satisfies the condition (C), and the critical values of φ are bounded below. Then $C_n(\varphi, \infty) \neq 0$ implies $C_q(\varphi, \infty) \cong \delta_{q,n} \mathcal{G}$.

Proof. Choose a real number

$$b < \inf\{\varphi(v) : v \in \mathbf{R}^n, \varphi'(v) = 0\}.$$

The set

$$C = \{v \in \mathbf{R}^n : \varphi(v) \geq b\}$$

is closed. Note that C is also connected, otherwise following from the condition (C) and a standard linking-type argument, φ will have a critical value $c \leq b$.

Let $a < b$. Since there is no critical value of φ in $[a, b]$, we get a continuous map $\eta : [0, 1] \times \varphi_b \rightarrow \varphi_b$, such that

- (i) $\eta(0, \cdot) = \mathbf{1}_{\varphi_b}$, $\eta(1, \varphi_b) \subset \varphi_a$,
- (ii) $\eta(t, v) = v$, for $(t, v) \in [0, 1] \times \varphi_a$,
- (iii) $\varphi(\eta(t, v))$ is nonincreasing in t .

From (iii), if $\varphi(v) < b$, then $\varphi(\eta(t, v)) < b$. Thus η maps $[0, 1] \times (\mathbf{R}^n \setminus C)$ into $\mathbf{R}^n \setminus C$. So φ_a is a strong deformation retractor of $\mathbf{R}^n \setminus C$, and we have

$$H_q(\mathbf{R}^n, \mathbf{R}^n \setminus C) \cong H_q(\mathbf{R}^n, \varphi_a), \quad \text{for } q = 0, 1, 2, \dots \tag{2.1}$$

We claim that C is compact: For otherwise, since C is closed and connected, by [7, Corollary VIII.3.4] we will have $H_n(\mathbf{R}^n, \mathbf{R}^n \setminus C) \cong 0$. But, since

$$a < b < \inf\{\varphi(v) : v \in \mathbf{R}^n, \varphi'(v) = 0\},$$

from (2.1) we get

$$C_n(\varphi, \infty) \cong H_n(\mathbf{R}^n, \varphi_a) \cong H_n(\mathbf{R}^n, \mathbf{R}^n \setminus C) \cong 0,$$

which is a contradiction.

Since C is compact, φ must be bounded from above. Moreover, by a result due to Zhong [18, Corollary 3.8], we know that

$$\varphi(v) \rightarrow -\infty, \quad \text{as } |v| \rightarrow \infty,$$

where $|v|$ is the Euclidean norm of v . Thus we can choose

$$a < b < \inf\{\varphi(v) : v \in \mathbf{R}^n, \varphi'(v) = 0\},$$

and $R > r > 0$, such that

$$\varphi_b \supset \mathbf{R}^n \setminus B_r \supset \varphi_a \supset \mathbf{R}^n \setminus B_R,$$

where $B_\rho = \{v \in \mathbf{R}^n : |v| \leq \rho\}$.

It is well known that φ_a is a strong deformation retractor of φ_b , thus

$$H_q(\varphi_b, \varphi_a) \cong 0, \quad \text{for } q = 0, 1, 2, \dots$$

Consider the following portion of the exact sequence of the triple $(\mathbf{R}^n, \varphi_b, \varphi_a)$

$$\begin{aligned} 0 &= H_q(\varphi_b, \varphi_a) \longrightarrow H_q(\mathbf{R}^n, \varphi_a) \\ &\xrightarrow{i_*} H_q(\mathbf{R}^n, \varphi_b) \xrightarrow{\partial_*} H_{q-1}(\varphi_b, \varphi_a) = 0 \end{aligned}$$

and that of the triple $(\mathbf{R}^n, \mathbf{R}^n \setminus B_r, \mathbf{R}^n \setminus B_R)$

$$0 = H_q(\mathbf{R}^n \setminus B_r, \mathbf{R}^n \setminus B_R) \longrightarrow H_q(\mathbf{R}^n, \mathbf{R}^n \setminus B_R) \\ \xrightarrow{k_*} H_q(\mathbf{R}^n, \mathbf{R}^n \setminus B_r) \xrightarrow{\partial_*} H_{q-1}(\mathbf{R}^n \setminus B_r, \mathbf{R}^n \setminus B_R) = 0,$$

where i_* , k_* are induced by inclusions. It follows that i_* and k_* are isomorphisms.

We also have the following commutative diagram:

$$\begin{array}{ccc} H_q(\mathbf{R}^n, \varphi_a) & \xrightarrow{i_*} & H_q(\mathbf{R}^n, \varphi_b) \\ \uparrow & \searrow l_* & \uparrow \\ H_q(\mathbf{R}^n, \mathbf{R}^n \setminus B_R) & \xrightarrow{k_*} & H_q(\mathbf{R}^n, \mathbf{R}^n \setminus B_r), \end{array}$$

where all the homomorphisms are induced by inclusions. Since i_* and k_* are isomorphisms, we deduce that l_* is also an isomorphism. Thus

$$C_q(\varphi, \infty) \cong H_q(\mathbf{R}^n, \varphi_a) \cong H_q(\mathbf{R}^n, \mathbf{R}^n \setminus B_r) \cong \delta_{q,n} \mathcal{G}.$$

This completes the proof. □

Lemma 2.2. *Let E be a Banach space. Assume that $\varphi \in C^1(E, \mathbf{R})$, the critical values of φ are bounded below. Then $C_0(\varphi, \infty) \neq 0$ implies $C_q(\varphi, \infty) \cong \delta_{q,0} \mathcal{G}$.*

Proof. If $C_0(\varphi, \infty) \neq 0$, then φ is bounded from below. Otherwise, for any

$$a < \inf\{\varphi(v) : v \in E, \varphi'(v) = 0\},$$

φ_a will not be empty. Since E is path-connected, by [8, Proposition 13.10] we have

$$C_0(\varphi, \infty) \cong H_0(E, \varphi_a) \cong 0,$$

a contradiction.

Thus, for a close to $-\infty$, we will have $\varphi_a = \emptyset$. So

$$C_q(\varphi, \infty) \cong H_q(E, \varphi_a) \cong H_q(E) \cong \delta_{q,0} \mathcal{G}.$$

This completes the proof. □

Proof of Theorem 1.1. By Proposition 2.2, we have a continuous function $\psi : X^- \rightarrow X^+$ and a C^1 -functional $\varphi : X^- \rightarrow \mathbf{R}$, such that

$$\varphi(v) = f(v + \psi(v)) = \min_{w \in X^+} f(v + w).$$

Moreover, φ satisfies the (PS) condition (thus the condition (C)), and the critical values of φ are bounded from below. Thus the critical groups $C_*(\varphi, \infty)$ are well defined.

If we choose $a \in \mathbf{R}$ such that the critical values of f and φ are bounded from below by a , then we have

$$C_q(f, \infty) = H_q(X, f_a), \quad C_q(\varphi, \infty) = H_q(X^-, \varphi_a).$$

Let

$$A := \{(v, \psi(v)) : v \in X^-\}, \quad B := \{(v, \psi(v)) : v \in \varphi_a\}.$$

From the definition of φ , if $(v, w) \in f_a$, then $v \in \varphi_a$. Thus we have the following diagram:

$$(X, f_a) \xrightarrow{h} (A, B) \xleftarrow{g} (X^-, \varphi_a),$$

where $g(v) = (v, \psi(v))$ and $h(v, w) = (v, \psi(v))$. Obviously g is a homeomorphism with inverse $g^{-1}(v, \psi(v)) = v$. We will show that h is a homotopic equivalence below. Thus passing to homology, g_* and h_* are isomorphisms, and we get

$$\begin{aligned} C_q(f, \infty) &= H_q(X, f_a) \cong H_q(A, B) \\ &\cong H_q(X^-, \varphi_a) = C_q(\varphi, \infty). \end{aligned} \tag{2.2}$$

The first conclusion of Theorem 1.1 follows. If $k = \dim X_- < \infty$ and $C_k(f, \infty) \neq 0$, then (2.2) results $C_k(\varphi, \infty) \neq 0$. By Lemma 2.1, we have $C_q(\varphi, \infty) \cong \delta_{q,k}\mathcal{G}$. So

$$C_q(f, \infty) \cong C_q(\varphi, \infty) \cong \delta_{q,k}\mathcal{G}.$$

This is our second conclusion.

Now we show that h is a homotopic equivalence. Note that (1.2) implies that f is convex on w , that is, for $v \in X^-$ and $w_1, w_2 \in X^+$,

$$f(v, (1-t)w_1 + tw_2) \leq (1-t)f(v, w_1) + tf(v, w_2), \quad 0 \leq t \leq 1.$$

Hence we can define $F : ([0, 1] \times X, [0, 1] \times f_a) \rightarrow (X, f_a)$,

$$F(t, (v, w)) = (v, (1-t)w + t\psi(v)).$$

Using the homotopy F , let $i : (A, B) \rightarrow (X, f_a)$ be the inclusion, it is easy to see that $h \circ i = \mathbf{1}_{(A, B)}$, $i \circ h \simeq \mathbf{1}_{(X, f_a)}$. So h is a homotopic equivalence. □

Proof of Theorem 1.2. By Proposition 2.2, we have a continuous function $\psi : X^- \rightarrow X^+$ and a C^1 -functional $\varphi : X^- \rightarrow \mathbf{R}$, such that

$$\varphi(v) = f(v + \psi(v)) = \max_{w \in X^+} f(v + w).$$

As in the proof of Theorem 1.1, the critical groups $C_*(\varphi, \infty)$ are well defined, and we can choose $a \in \mathbf{R}$, such that

$$C_q(f, \infty) = H_q(X, f_a), \quad C_q(\varphi, \infty) = H_q(X^-, \varphi_a).$$

For $v \in \{z \in X^- : \varphi(z) > a\}$, $w \in X^+ \setminus \{\psi(v)\}$, let

$$g(t) = f(v, w + t(w - \psi(v))), \quad t \geq 0.$$

By (1.3), it is easy to see that $\lim_{t \rightarrow \infty} g(t) = -\infty$. Thus if $f(v, w) > a$, there exists a $T(v, w) > 0$, such that

$$f(v, w + T(v, w)(w - \psi(v))) = a.$$

Since for all $\xi \in X^+$, we have $\langle \nabla f(v, \psi(v)), \xi \rangle = 0$. Hence

$$\begin{aligned} &-\frac{\partial}{\partial t} f(v, w + t(w - \psi(v))) \\ &= -\langle \nabla f(v, w + t(w - \psi(v))), w - \psi(v) \rangle \\ &= \frac{\langle \nabla f(v, w + t(w - \psi(v))) - \nabla f(v, \psi(v)), w + t(w - \psi(v)) - \psi(v) \rangle}{-(1+t)} \\ &\geq m \|w - \psi(v)\|^2 (1+t) > 0, \quad t \geq 0. \end{aligned}$$

Now by the implicit function theorem, $T(v, w)$ is unique and continuous with respect to $(v, w) \in X \cap f^{-1}(a, +\infty)$.

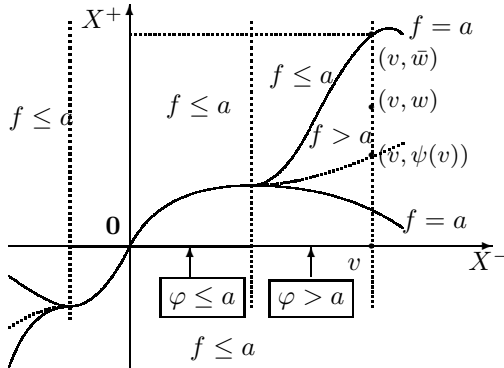
Let $A = (\varphi_a \times X^+) \cup \{(v, w) : \varphi(v) > a, w \neq \psi(v)\}$. Obviously, the function T can be extended to $\hat{T} : A \rightarrow \mathbf{R}$ continuously in the following way:

$$\hat{T}(v, w) = \begin{cases} 0, & \text{if } f(v, w) \leq a, \\ T(v, w), & \text{if } f(v, w) > a. \end{cases}$$

Now we define a homotopy $F : [0, 1] \times A \rightarrow A$,

$$F(t, (v, w)) = (v, (1-t)w + t(w + \hat{T}(v, w)(w - \psi(v)))).$$

Clearly, $F(1, A) = f_a$, $F(1, \cdot)$ is a homotopic equivalence between A and f_a . See the following figure, where $\bar{w} = w + \hat{T}(v, w)(w - \psi(v))$.



Let $G : A \rightarrow (\varphi_a \times X^+) \cup (X^- \times S)$ be defined as

$$G(v, w) = (v, w - \psi(v)),$$

where $S = \{w \in X^+ : w \neq 0\}$. Then G is a homeomorphism.

Now by the Künneth formula we have

$$\begin{aligned} C_*(f, \infty) &= H_*(X^- \times X^+, f_a) \cong H_*(X^- \times X^+, A) \\ &\cong H_*(X^- \times X^+, (\varphi_a \times X^+) \cup (X^- \times S)) \\ &= H_*((X^-, \varphi_a) \times (X^+, S)) \end{aligned}$$

$$\begin{aligned} &\cong H_*(X^-, \varphi_a) \otimes H_*(X^+, S) \\ &\cong H_{*-j}(X^-, \varphi_a) = C_{*-j}(\varphi, \infty). \end{aligned} \tag{2.3}$$

Here we use the fact that $H_q(X^+, S) \cong H_q(\mathbf{R}^j, \mathbf{R}^j \setminus \{0\}) = \delta_{q,j} \mathcal{G}$.

From (2.3), if $C_j(f, \infty) \neq 0$, then $C_0(\varphi, \infty) \neq 0$. By Lemma 2.2,

$$C_q(f, \infty) \cong C_{q-j}(\varphi, \infty) \cong \delta_{q-j,0} \mathcal{G} = \delta_{q,j} \mathcal{G}.$$

This completes the proof. □

Since the condition (C) is weaker than the (PS) condition, we wish that Theorems 1.1 and 1.2 can still hold under the condition (C). Checking the proof of Theorems 1.1 and 1.2, it suffices to ensure that φ satisfies the condition (C). Unfortunately, in the setting of Proposition 2.2, if f satisfies the condition (C), we can not deduce that φ also satisfies the condition (C). However, if we assume further that

$$\|f'(u)\| \leq C(1 + \|u\|), \tag{2.4}$$

then Zou–Liu [19, Lemma 2.5] has proved that if f satisfies the condition (C), then φ does also satisfy the condition (C). Thus we have

Corollary 2.1. *Assume that $f \in C^1(X, \mathbf{R})$ satisfies (2.4). If we replace the (PS) condition with the condition (C), then all the conclusions of Theorems 1.1 and 1.2 still hold.*

Finally, we state a version of the Morse inequality, see [5, Theorem I.4.3].

Theorem 2.1 (Morse inequality). *Suppose that $f \in C^1(X, \mathbf{R})$ satisfies the condition (C), has only isolated critical points, and the critical values of f are bounded from below. Then we have*

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1+t)Q(t), \tag{2.5}$$

where $M_q = \sum_{u \in K} \text{rank } C_q(f, u)$, $\beta_q = \text{rank } C_q(f, \infty)$, and Q is a formal series with nonnegative coefficients.

Corollary 2.2. *Under the assumption of Theorem 2.1, if $C_\ell(f, \infty) \neq 0$, then f has a critical point u with $C_\ell(f, u) \neq 0$.*

3. Applications

In this section, as applications of our abstract result, we consider the following semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = p(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded open domain with smooth boundary $\partial\Omega$ and $p : \mathbf{R} \rightarrow \mathbf{R}$ is a C^1 -function satisfying $p(0) = 0$, thus (3.1) admits a trivial solution $u = \mathbf{0}$. Our main goal is to find nontrivial solutions for (3.1).

We are interested in the following case

$$p_0 := p'(0), \quad p_\infty := \lim_{|u| \rightarrow \infty} p'(u),$$

that is, the problem (3.1) is asymptotically linear at both zero and infinity. Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ be the distinct eigenvalues of $(-\Delta, H_0^1(\Omega))$, where $H_0^1(\Omega)$ is the well known Sobolev space with inner product

$$\langle u, v \rangle = \int_{\Omega} \nabla u \nabla v \, dx$$

and corresponding norm $\|u\| = \langle u, u \rangle^{1/2}$. Of course, we will focus on the resonance case: $p_\infty = \lambda_\ell$ is an eigenvalue of $(-\Delta, H_0^1(\Omega))$.

Let $q(t) := p(t) - p_\infty t$ and $Q(t) := \int_0^t q(s) \, ds$ be the primitive of q . Then we have

Theorem 3.1. *Suppose $p_0 < \lambda_1 < p_\infty = \lambda_\ell, p'(t) \leq \beta < \lambda_{\ell+1}$ for some $\beta \in \mathbf{R}$. If*

$$\frac{Q(t)}{|t|} \rightarrow +\infty, \quad \text{as } |t| \rightarrow \infty, \tag{q}$$

then (3.1) has at least four nontrivial solutions.

Proof. It suffices to find four nonzero critical points of the following functional $f : X = H_0^1(\Omega) \rightarrow \mathbf{R}$,

$$f(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 - p_\infty u^2) \, dx - \int_{\Omega} Q(u) \, dx,$$

Obviously, $f \in C^2(X, \mathbf{R})$.

Since $p_0 < \lambda_1$, the zero function $\mathbf{0}$ is a local minimizer of f , thus

$$C_q(f, \mathbf{0}) \cong \delta_{q,0} \mathcal{G}. \tag{3.2}$$

Now we decompose $X = X^- \oplus X^+$ according to $p_\infty = \lambda_\ell$. More precisely, we set

$$X^- = \bigoplus_{i=1}^{\ell} \ker(-\Delta - \lambda_i), \quad X^+ = (X^-)^\perp = \overline{\bigoplus_{i \geq \ell+1} \ker(-\Delta - \lambda_i)}.$$

It has been shown in [14, Sec. 2] that (q) implies that f satisfies the (PS) condition, and

$$\begin{aligned} f(u) &\rightarrow -\infty, & \text{as } u \in X^-, \|u\| &\rightarrow \infty, \\ f(u) &\rightarrow +\infty, & \text{as } u \in X^+, \|u\| &\rightarrow \infty. \end{aligned}$$

Hence by [3, Proposition 3.8] we have $C_k(f, \infty) \neq 0$, where $k = \dim X^-$.

Since $p'(t) \leq \beta < \lambda_{\ell+1}$, for $v \in X^-$ and $w_1, w_2 \in X^+$ we have

$$\langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq m \|w_1 - w_2\|^2,$$

where $m = 1 - \beta(\lambda_{\ell+1})^{-1} > 0$. The details can be found in [4, Sec. 2]. Note that $C_k(f, \infty) \neq 0$, by Theorem 1.1 we obtain

$$C_q(f, \infty) \cong \delta_{q,k} \mathcal{G}. \tag{3.3}$$

By Corollary 2.2, we know that f has a critical point u_1 with $C_k(f, u_1) \neq 0$. Moreover, the condition $p'(t) \leq \beta < \lambda_{\ell+1}$ implies that

$$(\text{Morse index of } f \text{ at } u_1) + (\text{nullity of } f \text{ at } u_1) \leq k.$$

Note that $C_k(f, u_1) \neq 0$, Proposition 2.1 results

$$(\text{Morse index of } f \text{ at } u_1) + (\text{nullity of } f \text{ at } u_1) = k$$

and

$$C_q(f, u_1) \cong \delta_{q,k} \mathcal{G}. \tag{3.4}$$

From (3.2) and (3.4), we know that $u_1 \neq \mathbf{0}$.

Since $p_0 < \lambda_1 < p_\infty$, by the mountain pass lemma and the truncated technique, as in [6, 17], we obtain another two critical point u_+ , u_- of mountain pass type, with

$$C_q(f, u_\pm) \cong \delta_{q,1} \mathcal{G}, \quad \pm u_\pm > 0.$$

Assume that $\mathbf{0}$, u_+ , u_+ and u_- are the only critical points of f . Then the Morse inequality (2.5) with $t = -1$ becomes

$$(-1)^0 + (-1)^1 \times 2 + (-1)^k = (-1)^k.$$

This is impossible. Thus f must have at least one more critical point. So (3.1) has at least four nontrivial solutions. □

Remark 3.1. For related results, see [4, 12]. It was assumed in [12] that q is bounded and satisfies the standard Landesman–Lazer condition; or for some $\alpha \in (0, 1)$, q satisfies

$$(q_1) \quad \exists c_1 > 0, \text{ such that } |q(t)| \leq c_1(1 + |t|^\alpha),$$

$$(q_2) \quad \frac{Q(t)}{|t|^{2\alpha}} \rightarrow +\infty, \text{ as } |t| \rightarrow \infty,$$

see [12, Theorem 1.1] and [12, Theorem 1.2] respectively.

Our Theorem 3.1 does not require (q_1) . Moreover, if $\alpha \geq \frac{1}{2}$, our condition (q) is weaker than (q_2) .

Similarly we have

Theorem 3.2. *Suppose $p_0 < \lambda_1 < p_\infty = \lambda_\ell$, $p'(t) \leq \beta < \lambda_{\ell+1}$ for some $\beta \in \mathbf{R}$. If*

$$\lim_{|t| \rightarrow \infty} [q(t)t - 2Q(t)] = -\infty. \tag{q-}$$

then (3.1) has at least four nontrivial solutions.

In fact, it has been shown in [10, Lemma 3.5] that, (q_-) implies that f satisfies the condition (C) and $C_k(f, \infty) \neq 0$. Obviously f satisfies (2.4) thus we can use Corollary 2.1 to obtain (3.3).

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