Bull. London Math. Soc. 36 (2004) 81–87 © 2004 London Mathematical Society
DOI: 10.1112/S0024609303002546

EXISTENCE OF SOLUTIONS FOR ASYMPTOTICALLY 'LINEAR' p-LAPLACIAN EQUATIONS

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Abstract

Under very general conditions, a proof is given, via Morse theory, of the existence of solutions for asymptotically 'linear' p-Laplacian equations, where the asymptotic limit may be greater than the second eigenvalue. The existence of nonzero solutions is also considered.

1. *Introduction*

In this paper, we consider the following Dirichlet problem for the p-Laplacian equations:

$$\begin{cases} -\Delta_p u = f\left(x, u\right), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and

$$-\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\nabla u), \qquad 1 < p < N,$$

is the p-Laplacian. If $p = 2$, then $-\Delta_2 = -\Delta$, the usual Laplacian operator.

We shall assume the following asymptotically 'linear' conditions on the nonlinearity f.

(f_1) $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, and

$$\lim_{|u|\to\infty} \frac{f(x,u)}{|u|^{p-2}u} = \lambda.$$

Let us denote by $\sigma(-\Delta_p)$ the spectrum of $-\Delta_p$ as it acts on the standard Sobolev space $W_0^{1,p}(\Omega)$; that is, the set of real numbers μ such that the problem

$$\begin{cases} -\Delta_p u = \mu|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{E}$$

has nonzero solutions. Then one of our main results reads as follows.

THEOREM 1.1. *If condition* (f_1) *holds and* $\lambda \notin \sigma(-\Delta_p)$, *then the problem* (P) *has a solution.*

Received 1 October 2002.

2000 *Mathematics Subject Classification* 35J65, 58E05.

The first author was supported by the National Natural Science Foundation of China and the Natural Science Foundation of Beijing (1022003). The second author was supported by the 973 National Natural Sciences Foundation of China and the Foundation of the Chinese Academy of Sciences.

When $p = 2$, this result is well known. In this case, since the structure of $\sigma(-\Delta)$ is well understood, one can decompose the space $W_0^{1,2}(\Omega)$ according to the location of λ related to $\sigma(-\Delta)$, and apply the saddle point theorem. For general $p > 1$, we do not have enough knowledge about $\sigma(-\Delta_p)$, and thus the above approach will not work. However, thanks to Perera's elegant work [**7**], where a new sequence of eigenvalues of $-\Delta_p$ was constructed via the Yang index [**8**] and homologically characterized, we can obtain enough information about the *critical groups at infinity* [**1**] of the energy functional associated to problem (P), and thus we can obtain a solution of (P) via Morse theory.

Recall that if there exists a function $\bar{f} \in L^{p/(p-1)}(\Omega)$ such that

$$|f(x,u) - \lambda|u|^{p-2}u| \leqslant \bar{f}(x), \qquad \forall (x,u) \in \Omega \times \mathbb{R},$$

then the existence of a solution for problem (P) has been obtained by Drábek and Robinson [**3**]. Our Theorem 1.1 does not need this assumption.

If we impose some further conditions on f near zero, then we can obtain a nonzero solution, as shown in Theorem 1.2.

(f_2) There exist $r > 0$ and $\nu \in (1,p)$, such that for $x \in \Omega$ and $0 < |u| \leqslant r$, the following statements hold:

$$f(x,u)u > 0, \quad \text{and} \quad \frac{1}{\nu} f(x,u)u \leqslant F(x,u) := \int_0^u f(x,s)\,ds.$$

THEOREM 1.2. *If conditions* (f_1) *and* (f_2) *hold and* $\lambda \notin \sigma(-\Delta_p)$, *then the problem* (P) *has a nonzero solution.*

Similarly, for condition (f_3), we have Theorem 1.3.

(f_3) There exist $a > 0$ and $\tau \in (1,p)$ such that

$$\lim_{|u|\to 0} \frac{f(x,u) - a|u|^{\tau-2}u}{|u|^{p-2}u} = 0. \tag{1.1}$$

THEOREM 1.3. *If conditions* (f_1) *and* (f_3) *hold, then the problem* (P) *has a nonzero solution.*

REMARK 1.4. In [**3**], Drábek and Robinson have not discussed the existence of nonzero solutions.

This paper is organized as follows. In Section 2, we recall some necessary results of Perera [**7**] concerning his new sequence of eigenvalues, and in Section 3 we prove our theorems.

2. *Eigenvalues constructed by the Yang index*

Following Perera [**7**], let (X, A), $A \subset X$, be a pair of closed symmetric subsets of the Banach space, and let $C(X, A)$ be its singular chain complex with $\mathbb{Z}_2$-coefficients. Denote by $T_\#$ the chain map of $C(X, A)$ induced by the antipodal map $T : x \mapsto -x$. We say that a q-chain c is *symmetric* if and only if $T_\#(c) = c$. The symmetric q-chains form a subgroup $C_q(X, A; T)$ of $C_q(X, A)$. In this way, by using the original boundary operator, we obtain a subcomplex $C(X, A; T)$ of $C(X, A)$. We denote by $H_q(X, A; T)$ the homology of $C(X, A; T)$, and $H_q(X; T) := H_q(X, \emptyset; T)$.

As in [**7**], and due to [**8**, Proposition 2.8], we can then define the *index homomorphism* $\nu_* : H_q(X; T) \longrightarrow \mathbb{Z}_2$, and we see that if $\nu_* H_\ell(X; T) = \mathbb{Z}_2$, then $\nu_* H_q(X; T) = \mathbb{Z}_2$ for $0 \leqslant q \leqslant \ell$. Therefore we can define the *Yang index* of X by

$$i_{\mathrm{Y}}(X) := \inf\{l \geqslant -1 : \nu_* H_{l+1}(X; T) = 0\},$$

taking $\inf \emptyset = \infty$.

Now we consider the eigenvalue problem (E). The eigenvalues of (E) are the critical values of the C^1-functional

$$I(u) = \int_\Omega |\nabla u|^p \, dx, \qquad u \in S = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^p \, dx = 1 \right\},$$

which satisfies the (PS) condition. We denote by $\mathcal{A}$ the class of closed symmetric subsets of S, we let

$$\mathcal{F}_l = \{A \in \mathcal{A} : i_{\mathrm{Y}}(A) \geqslant l - 1\},$$

and we set

$$\lambda_l := \inf_{A \in \mathcal{A}} \sup_{u \in A} I(u).$$

Proposition 2.1 [**7**, Proposition 3.1]. *λ_l is an eigenvalue of* (E), $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_l \leqslant \lambda_{l+1} \leqslant \ldots$, *and $\lambda_l \to +\infty$ as $l \to \infty$.*

Using the above sequence of eigenvalues, and letting

$$I_\lambda(u) = \frac{1}{p} \int_\Omega (|\nabla u|^p - \lambda |u|^p) \, dx, \qquad u \in W_0^{1,p}(\Omega),$$

we have the following proposition.

Proposition 2.2 [**7**, Proposition 1.1]. *If $\lambda \in (\lambda_l, \lambda_{l+1}) \setminus \sigma(-\Delta_p)$, then $\mathbf{0}$ is the only critical point of I_λ, with $C_l(I_\lambda, \mathbf{0}) \neq 0$.*

Let us recall some notions in Morse theory, which have been used in the statement of Proposition 2.2. Let φ be a C^1-functional defined on a Banach space E; then the *critical group* of φ at an isolated critical point u with $\varphi(u) = c$ is defined by

$$C_q(\varphi, u) := H_q(\varphi_c, \varphi_c \setminus \{u\}; \mathcal{G}), \qquad q \in \mathbb{N}_0 := \{0, 1, 2, \ldots\},$$

where H_q is the singular relative homology with coefficients in an Abelian group $\mathcal{G}$ and $\varphi_c := \varphi^{-1}(-\infty, c]$; see [**2**] for the details. The *critical groups at infinity* of φ were introduced by Bartsch and Li [**1**] as

$$C_q(\varphi, \infty) := H_q(E, \varphi_\alpha; \mathcal{G}), \qquad q \in \mathbb{N}_0, \tag{2.1}$$

if φ satisfies the (PS) condition and the critical values of φ are bounded from below by some $\alpha > -\infty$. Note that by the deformation lemma, the right-hand side of (2.1) does not depend on the choice of α.

In the proofs of our theorems we shall use the following result.

Proposition 2.3 (see [**1**]). *Suppose that $\varphi \in C^1(E, \mathbb{R})$ satisfies the* (PS) *condition. Then the following statements hold.*

(1) *If for some* $k \in \mathbb{N}_0$ *there exists* $C_k(\varphi,\infty) \neq 0$, *then* φ *has a critical point* u *with* $C_k(\varphi,u) \neq 0$.

(2) *Let* $\mathbf{0}$ *be an isolated critical point of* φ. *If for some* $k \in \mathbb{N}_0$ *there exists* $C_k(\varphi,\infty) \neq C_k(\varphi,\mathbf{0})$, *then* φ *has a nonzero critical point.*

3. *Proofs of the main results*

Define the following functional $\Phi : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ on $W_0^{1,p}(\Omega)$:

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F(x,u)\,dx, \qquad F(x,t) := \int_0^t f(x,s)\,ds.$$

Then, by condition (f$_1$), Φ is of class C^1 and the critical points of Φ are solutions of problem (P). We may assume that Φ has only finitely many critical points. Since $\lambda \notin \sigma(-\Delta_p)$, it is also well known that Φ satisfies the (PS) condition.

Dual to the argument of Perera [**7**], we shall construct a perturbed functional $\tilde{\Phi}$ that has the same critical points as Φ, and is equal to I_λ outside a big ball, and equal to Φ inside a smaller ball. Let $\|\cdot\|$ denote the norm, both in $W_0^{1,p}(\Omega)$ and in its dual space $W^{-1,p'}(\Omega)$.

Lemma 3.1. *There exist* $R > 0$ *and* $\tilde{\Phi} \in C^1(W_0^{1,p}(\Omega),\mathbb{R})$, *such that*

$$\tilde{\Phi}(u) = \begin{cases} I_\lambda(u), & \text{for } \|u\| \geqslant \sqrt[p]{2}R, \\ \Phi(u), & \text{for } \|u\| \leqslant R. \end{cases} \tag{3.1}$$

Moreover, $\varepsilon := \inf_{\|u\|\geqslant R} \|\tilde{\Phi}'\| > 0$. *Thus the critical points of* $\tilde{\Phi}$ *are all critical points of* Φ.

Proof. Let

$$g(x,t) = f(x,t) - \lambda |t|^{p-2}t,$$
$$G(x,t) = \int_0^t g(x,s)\,ds,$$
$$\Psi(u) = \int_\Omega G(x,u)\,dx.$$

Thus $\Phi = I_\lambda - \Psi$. Since $\lambda \notin \sigma(-\Delta_p)$, and I_λ satisfies the (PS) condition and has no critical points on the unit sphere in $W_0^{1,p}(\Omega)$, we see that $\delta := \inf_{\|u\|=1} \|I'_\lambda\| > 0$. By homogeneity, $\inf_{\|u\|=R} \|I'_\lambda\| = R^{p-1}\delta$. It follows from condition (f$_1$) that

$$\sup_{\|u\|=R} |\Psi| = o\,(R^p) \quad \text{and} \quad \sup_{\|u\|=R} \|\Psi'\| = o(R^{p-1}) \tag{3.2}$$

as $R \to \infty$. Since $\Phi' = I'_\lambda - \Psi'$, we have

$$\inf_{\|u\|=R} \|\Phi'\| \geqslant R^{p-1}\,(\delta + o\,(1)) > 0 \tag{3.3}$$

for all sufficiently large $R > 0$. We may require that all the critical points of Φ be contained in $\{u \in W_0^{1,p}(\Omega) : \|u\| \leqslant R\}$.

We take a smooth function $\varphi : [0,\infty) \longrightarrow [0,1]$ with $|\varphi'(t)| \leqslant 1$, such that

$$\varphi(t) = \begin{cases} 0, & \text{for } 0 \leqslant t \leqslant 1, \\ 1, & \text{for } t \geqslant 2. \end{cases}$$

Then we set

$$\tilde{\Phi}(u) = \Phi(u) + \varphi\left(\frac{\|u\|^p}{R^p}\right)\Psi(u).$$

It is easy to see that (3.1) holds. Using (3.2), (3.3) and $|\varphi'(t)| \leqslant 1$, we obtain

$$\inf_{\|u\|=R} \|\tilde{\Phi}'\| \geqslant R^{p-1}(\delta + o(1)) > 0.$$

From this it is easy to see that the conclusion of this lemma holds true. □

Proof of Theorem 1.1. Since $\lambda \notin \sigma(-\Delta_p)$, we may assume that

$$\lambda \in (\lambda_\ell, \lambda_{\ell+1}) \setminus \sigma(-\Delta_p)$$

for some $\ell \in \mathbb{N}_0$. Therefore, by Propositions 2.2 and 2.3, the zero function $\mathbf{0}$ is the only critical point of I_λ and

$$C_\ell(I_\lambda, \infty) = C_\ell(I_\lambda, \mathbf{0}) \neq 0.$$

Since $\tilde{\Phi} = I_\lambda$ outside a big ball B, if we choose

$$\alpha < \min\left\{\inf_B \tilde{\Phi}, \inf_B I_\lambda\right\}$$

in the definition of critical groups at infinity (2.1), we easily see that

$$C_\ell(\tilde{\Phi}, \infty) = C_\ell(I_\lambda, \infty) \neq 0.$$

By Proposition 2.3, $\tilde{\Phi}$ has at least a critical point u with $C_\ell(\tilde{\Phi}, u) \neq 0$. By Lemma 3.1, u is also a critical point of Φ, with $C_\ell(\Phi, u) \neq 0$, since $\Phi = \tilde{\Phi}$ in a small neighborhood of u. □

To prove the existence of nonzero solutions, we need to compute the critical groups at zero. First, we set further conditions.

(f) $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, and there exist some $C > 0$ and $\theta \in (p, p^*)$, such that

$$|f(x,u)| \leqslant C(1 + |u|^{\theta-1}), \qquad (x,u) \in \Omega \times \mathbb{R}.$$

PROPOSITION 3.2 [**4**, Proposition 2.1]. *If conditions* (f_2) *and* (f) *hold, then* $C_k(\Phi, \mathbf{0}) = 0$, *for all* $k \in \mathbb{N}_0$.

REMARK 3.3. In the case where $p = 2$, related results can be found in [**5**, **6**].

LEMMA 3.4. *Assume that conditions* (f) *and* (f_3) *hold; then* $C_k(\Phi, \mathbf{0}) = 0$, *for all* $k \in \mathbb{N}_0$.

Proof. From (1.1), it is easy to see that

$$\lim_{|u|\to 0} \frac{f(x,u)}{|u|^{\tau-2}u} = a. \tag{3.4}$$

It follows from condition (f) and (3.4) that

$$F(x,u) \geqslant C_0|u|^\tau - C_1|u|^\theta, \qquad \forall (x,u) \in \Omega \times \mathbb{R}.$$

Hence for $u \in W_0^{1,p}(\Omega) \setminus \{\mathbf{0}\}$ and $s > 0$, we have

$$\begin{aligned}\Phi(su) &= \frac{s^p}{p}\int_\Omega |\nabla u|^p\,dx - \int_\Omega F(x,su)\,dx \\ &\leqslant \frac{s^p}{p}\|u\|^p - C_0 s^\tau |u|_\tau^\tau + C_1 s^\theta |u|_\theta^\theta;\end{aligned}$$

here, we use $|\cdot|_\gamma$ to denote the standard L^γ-norm. Since in the one-dimensional space span$\{u\}$, all norms are equivalent, for the given $u \neq \mathbf{0}$ there exists $s_0 = s_0(u) \in (0,1)$, such that

$$\Phi(su) < 0, \qquad \text{for } s \in (0, s_0). \tag{3.5}$$

On the other hand, from (1.1) we have

$$\lim_{|u|\to 0} \frac{\tau F(x,u) - f(x,u)u}{|u|^p} = 0.$$

Note that

$$\left| F(x,u) - \frac{1}{\tau} f(x,u)u \right| \leqslant C(1 + |u|^\theta),$$

so we have

$$\int_\Omega \left(F(x,u) - \frac{1}{\tau} f(x,u)u \right) dx = o(\|u\|^p), \qquad \text{as } \|u\| \to 0.$$

Thus by a direct computation we obtain

$$\frac{1}{\tau}\left.\frac{d}{ds}\right|_{s=1} \Phi(su) = \Phi(u) + \left(\frac{1}{\tau} - \frac{1}{p}\right)\|u\|^p + o(\|u\|^p), \qquad \text{as } \|u\| \to 0.$$

From this, there exists $\rho > 0$, such that

$$\left.\frac{d}{ds}\right|_{s=1} \Phi(su) > 0, \qquad \forall u \in \Phi^{-1}[0,+\infty) \cap B_\rho \setminus \{\mathbf{0}\}, \tag{3.6}$$

where $B_\rho := \{u \in W_0^{1,p}(\Omega) : \|u\| \leqslant \rho\}$.

Next we follow the arguments in the proof of [**4**, Proposition 2.1]. It follows by the monotonicity arguments from (3.6) that

$$\Phi(su) < 0, \qquad \text{for } s \in (0,1),\ u \in \Phi^{-1}(-\infty, 0) \cap B_\rho. \tag{3.7}$$

From (3.5), (3.6) and (3.7), for $u \in B_\rho$, if $\Phi(u) > 0$, then there exists a unique $T(u) \in (0,1)$ such that $\Phi(T(u)) = 0$,

$$\Phi(su) < 0 \text{ for } s \in (0, T(u)) \quad \text{and} \quad \Phi(su) > 0 \text{ for } s \in (T(u), 1). \tag{3.8}$$

If $\Phi(u) \leqslant 0$, we set $T(u) = 1$. It follows from (3.6), (3.8) and the implicit function theorem, that $T : B_\rho \longrightarrow [0,1]$ is continuous in u. Now we define $\eta : [0,1] \times B_\rho \longrightarrow B_\rho$ by

$$\eta(s,u) = (1-s)u + sT(u)u, \qquad (s,u) \in [0,1] \times B_\rho.$$

It is easy to see that η is a continuous deformation from $(B_\rho, B_\rho \setminus \{\mathbf{0}\})$ to $(\Phi_0 \cap B_\rho, \Phi_0 \cap B_\rho \setminus \{\mathbf{0}\})$. Thus

$$C_k(\Phi, \mathbf{0}) = H_k(\Phi_0 \cap B_\rho, \Phi_0 \cap B_\rho \setminus \{\mathbf{0}\}) \cong H_k(B_\rho, B_\rho \setminus \{\mathbf{0}\}) \cong 0, \quad \forall k \in \mathbb{N}_0,$$

since $B_\rho \setminus \{\mathbf{0}\}$ is contractible. ☐

Proof of Theorem 1.2. We may assume that $\mathbf{0}$ is a critical point of Φ; otherwise we already have a nonzero solution, by Theorem 1.1. By Proposition 3.2, under conditions (f_1) and (f_2), we know that

$$C_k(\Phi, \mathbf{0}) = 0, \qquad \forall k \in \mathbb{N}_0.$$

However, the critical point u obtained in Theorem 1.1 satisfies $C_\ell(\Phi, u) \neq 0$. Thus $u \neq \mathbf{0}$. □

Proof of Theorem 1.3. This is similar to the proof of Theorem 1.2, but instead of Proposition 3.2, we now use Lemma 3.4 to compute the critical groups at zero, and also to obtain $C_*(\Phi, \mathbf{0}) = 0$. □

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