On superlinear $p(x)$-Laplacian equations in R^N ☆

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Abstract

We consider the $p(x)$-Laplacian equations in R^N. The nonlinearity is superlinear but dose not satisfy the Ambrosetti-Rabinowitz type condition. We obtain ground states of the equations, improving a recent result of Fan [J. Math, Anal. Appl. 341 (2008), 103–119]. We also establish a Bartsch-Wang type compact embedding theorem for variable exponent spaces. Then, a multiplicity result for the equations is proved for odd nonlinearity.

Key words: $p(x)$-Laplacian, Superlinear problems, Cerami sequences, Fountain Theorem
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1. Introduction

In this paper, we consider the following $p(x)$-Laplacian equation in R^N,

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + V(x)\,|u|^{p(x)-2}\,u = f(x,u), & \text{in } \mathsf{R}^N, \\ u \in W^{1,p(x)}(\mathsf{R}^N). \end{cases} \tag{1.1}$$

Here $W^{1,p(x)}(\mathsf{R}^N)$ is the variable exponent Sobolev space. For the sake of simplicity, we say that $g : \mathsf{R}^N \to \mathsf{R}$ is periodic if $g(x+z) = g(x)$ for any $x \in \mathsf{R}^N$ and $z \in \mathsf{Z}^N$. We denote $a \ll b$ provided $\inf\left\{b(x) - a(x) \,\middle|\, x \in \mathsf{R}^N\right\} > 0$. Now, we make the following assumptions on the functions p, V, and f.

(p) The function $p : \mathsf{R}^N \to \mathsf{R}$ is Lipschitz continuous and

$$1 < p_- := \inf_{x\in\mathsf{R}^N} p(x) \le \sup_{x\in\mathsf{R}^N} p(x) =: p_+ < N. \tag{1.2}$$

(V) $V \in C(\mathsf{R}^N)$, $0 < V_- \le V_+ < +\infty$.

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(f_1) $f \in C\big(\mathbb{R}^N \times \mathbb{R}\big)$ satisfies

$$\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{q(x)-1}} = 0, \qquad \lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{p_+}} = +\infty \tag{1.3}$$

uniformly in $x \in \mathbb{R}^N$, for some $q \in L^\infty(\mathbb{R}^N)$ such that $p_+ \ll q \ll p^*$, where

$$p^*(x) = \frac{Np(x)}{N - p(x)}, \qquad F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s.$$

(f_2) $f(x,t) = o\big(|t|^{p(x)-1}\big)$ as $t \to 0$, uniformly in $x \in \mathbb{R}^N$.

(f_3) There exists $\theta \geq 1$ such that $\theta\mathcal{F}(x,t) \geq \mathcal{F}(x,st)$ for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0,1]$, where $\mathcal{F}(x,t) = f(x,t)t - p_+F(x,t)$.

The second limit in (1.3) is a consequence of

$$\lim_{|t|\to\infty} \frac{f(x,t)t}{|t|^{p_+}} = +\infty.$$

For this reason, we say that our problem (1.1) is superlinear. The condition (f_3) is originally due to Jeanjean [15] in the case $p(x) \equiv 2$, and then was used in Liu and Li [21] for p-Laplacian equations in bounded domain. It is known that (f_3) is weaker than the condition that

$$\text{for each } x \in \mathbb{R}^N,\ \frac{f(x,t)}{|t|^{p_+-1}} \text{ is an increasing function of } t \text{ in } \mathbb{R}\setminus\{0\}, \tag{1.4}$$

see Liu and Li [21, Proposition 2.3] for a proof; see also Jeanjean and Tanaka [16, Lemma 2.1] for the case $p(x) \equiv 2$.

By conditions (f_1) and (f_2), there exists a constant $C > 0$ such that

$$|F(x,t)| \leq C\big(|t|^{p(x)} + |t|^{p^*(x)}\big), \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Hence the energy functional $\Phi : W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\Phi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \Big(|\nabla u|^{p(x)} + V(x)\,|u|^{p(x)}\Big)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \tag{1.5}$$

is well defined and of class C^1. The derivative of Φ is given by

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} \Big(|\nabla u|^{p(x)-2}\,\nabla u \cdot \nabla v + V(x)\,|u|^{p(x)-2}\,uv\Big)\,\mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x \tag{1.6}$$

for $v \in W^{1,p(x)}(\mathbb{R}^N)$. Therefore, the critical points of Φ are weak solutions of (1.1). Now we are ready to state our first result.

Theorem 1.1. *Suppose that the conditions (p), (V), (f_1), (f_2) and (f_3) hold. If p, V are periodic, and $f(\cdot,t)$ is periodic for all $t \in \mathbb{R}$, then the problem* (1.1) *has a* ground state, *i.e. a nontrivial solution v such that*

$$\Phi(v) = \inf\Big\{\Phi(u)\ \Big|\ u \in W^{1,p(x)}(\mathbb{R}^N)\setminus\{\mathbf{0}\},\ \Phi'(u) = 0\Big\}.$$

This theorem improves a recent result of Fan [9, Theorem 1.2]. In that paper, to obtain a ground state, in addition to (p), (V), (f_1), (f_2) and the periodic assumption, the author assumed the condition (1.4) and the well known Ambrosetti-Rabinowitz condition (see [1]), that is,

(AR) there exists $\beta > p_+$ such that

$$0 < \beta F(x,t) \leq f(x,t)t, \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

In our Theorem 1.1 the condition (AR) is completely removed, and our assumption (f_3) is weaker than the monotonicity condition (1.4).

Remark 1.2. Recently, for the case that $p(x)$ is a constant $p \in (1, N)$, similar result has been obtained in Liu [19, Theorem 1.1]. Although the present work is motivated by [19], which in turn is influenced by [16, 21], the variable exponent case considered here is more difficult and more delicate. For example, in Zang [27], where a superlinear $p(x)$-Laplacian equation on a bounded domain $\Omega \subset \mathbb{R}^N$ is studied in the spirit of [21], to remove the condition (AR), the following condition is required:

$(\mathcal{F})$ There exists $\theta \geq 1$ such that for any $\lambda, \mu \in [p_-, p_+]$ and $s \in [0, 1]$ there holds

$$\theta \mathcal{F}_\lambda(x,t) \geq \mathcal{F}_\mu(x, st), \qquad (x,t) \in \Omega \times \mathbb{R},$$

where $\mathcal{F}_\lambda(x,t) = f(x,t)t - \lambda F(x,t)$.

Obviously, our condition (f_3) is weaker than $(\mathcal{F})$ above. In the papers cited above, the main difficulty is that: due to the absence of the condition (AR), the variational functional Φ may possess unbounded (PS) sequences. To overcome this difficulty, the Cerami sequences are employed.

Next, we assume that the potential V satisfies the following condition:

(V_1) $V \in C(\mathbb{R}^N)$, $V_- > 0$, $\mu\big(V^{-1}(-\infty, M]\big) < +\infty$ for all $M \in \mathbb{R}$.

Here μ is the Lebesgue measure on $\mathbb{R}^N$. Note that if $V \in C\big(\mathbb{R}^N, (0 + \infty)\big)$ is coercive, namely

$$\lim_{|x| \to \infty} V(x) = +\infty,$$

then (V_1) is satisfied. The second result of this paper is the following theorem.

Theorem 1.3. *Suppose that the conditions* (p), (V_1), (f_1), (f_2) *and* (f_3) *hold. Then:*

(i) *the problem* (1.1) *has a nontrivial solution.*
(ii) *if in addition* f *is odd in* t, *that is* $f(x,-t) = -f(x,t)$ *for all* $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, *then* (1.1) *has a sequence of solutions* $\{u_n\}$ *such that* $\Phi(u_n) \to +\infty$.

Remark 1.4. For more existence and multiplicity results on $p(x)$-Laplacian equation in $\mathbb{R}^N$, we refer to Fan and Han [10]. In [10, Theorem 3.2], similar results are obtained for the case $V(x) \equiv 1$, which is not included in our Theorem 1.3. They assumed that f satisfies the Ambrosetti-Rabinowitz condition (AR) and that

$$|f(x,t)| \leq \sum_{i=1}^{m} b_i(x)|t|^{q_i(x)-1}, \tag{1.7}$$

where the weights b_i lie in some suitable variable Lebesgue space, so that the functional Φ satisfies the (PS) condition. Our condition (V_1) also implies that Φ satisfies some kind of compactness condition, but here we do not require the Ambrosetti-Rabinowitz condition (AR) and the growth condition (1.7).

The variable exponent variational problems appear in a lot of applications, see [18, 24]. In recent years, such problems have attracted an increasing attention, we mention [3, 10], and also the survey papers [4, 7, 25] for the advances and references in this area.

2. Variable exponent Sobolev space

In this section we recall some results on variable exponent Sobolev spaces. The reader is refered to [11, 13] and references therein for more details.

Let $p \in L^{\infty}(\mathbb{R}^N)$, $p_- > 1$. The variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^N)$ is defined by

$$L^{p(x)}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \,\middle|\, u \text{ is measurable and } \int_{\mathbb{R}^N} |u|^{p(x)} \, \mathrm{d}x < \infty \right\}$$

endowed with the norm

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 \,\middle|\, \int_{\mathbb{R}^N} \left| \frac{u}{\lambda} \right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Then we define the variable exponent Sobolev space

$$W^{1,p(x)}(\mathbb{R}^N) = \left\{ u \in L^{p(x)}(\mathbb{R}^N) \,\middle|\, |\nabla u| \in L^{p(x)}(\mathbb{R}^N) \right\}$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

When V satisfies (V), the norm

$$\|u\| = \inf \left\{ \lambda > 0 \,\middle|\, \int_{\mathbb{R}^N} \left(\left| \frac{\nabla u}{\lambda} \right|^{p(x)} + V(x) \left| \frac{u}{\lambda} \right|^{p(x)} \right) \mathrm{d}x \le 1 \right\} \tag{2.1}$$

is equivalent to the norm $\|\cdot\|_{1,p(x)}$. With these norms, the spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ are separable reflexive Banach spaces, see [13] for the details.

Proposition 2.1. *The functional* $\psi : W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ *defined by*

$$\psi(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + V(x) \, |u|^{p(x)} \right) \mathrm{d}x, \tag{2.2}$$

has the following properties:

(i) *If* $\|u\| \ge 1$, *then* $\|u\|^{p-} \le \psi(u) \le \|u\|^{p+}$.
(ii) *If* $\|u\| \le 1$, *then* $\|u\|^{p+} \le \psi(u) \le \|u\|^{p-}$.

In particular, if $\|u\| = 1$ *then* $\psi(u) = 1$; *moreover,* $\|u_n\| \to 0$ *if and only if* $\psi(u_n) \to 0$.

Remark 2.2. For the functional $\rho : L^{p(x)}(\mathbb{R}^N) \to \mathbb{R}$,

$$u \longmapsto \int_{\mathbb{R}^N} |u|^{p(x)} \, \mathrm{d}x,$$

we have similar results. For example, $|u_n|_{p(x)} \to 0$ if and only if $\rho(u_n) \to 0$; and, as a consequence of the corresponding (i) and (ii), we have

$$|u|_{p(x)} \le \left(\int_{\mathbb{R}^N} |u|^{p(x)} \, \mathrm{d}x \right)^{1/p_+} + \left(\int_{\mathbb{R}^N} |u|^{p(x)} \, \mathrm{d}x \right)^{1/p_-}. \tag{2.3}$$

For $p \in L^{\infty}(\mathbb{R}^N)$ with $p_- > 1$, let $p' : \mathbb{R}^N \to \mathbb{R}$ be such that

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \qquad \text{a.e. } x \in \mathbb{R}^N.$$

We have the following generalized Hölder inequality.

Proposition 2.3 ([23, p.9]). *For any $u \in L^{p(x)}(\mathbb{R}^N)$, $v \in L^{p'(x)}(\mathbb{R}^N)$, we have*

$$\left| \int_{\mathbb{R}^N} uv \, \mathrm{d}x \right| \leq 2 \, |u|_{p(x)} \, |v|_{p'(x)} . \tag{2.4}$$

Proposition 2.4 ([11, Theorems 1.1, 1.3]). *Let $p : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz continuous and satisfy (1.2), $q : \mathbb{R}^N \to \mathbb{R}$ be a measurable function.*

(i) *If $p \leq q \leq p^*$, then there is a continuous embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$.*
(ii) *If $p \leq q \ll p^*$, then there is a compact embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}_{\mathrm{loc}}(\mathbb{R}^N)$.*

We also need the following variable exponent generalization of the Lions Lemma [26, Lemma I.1]. For $r > 0$ and $y \in \mathbb{R}^N$ we denote by $B_r(y)$ the open ball in $\mathbb{R}^N$ with center y and radius r.

Proposition 2.5 ([14, Lemma 3.1]). *Suppose $p : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and satisfies (1.2). If $\{u_n\}$ is a bounded sequence in $W^{1,p(x)}(\mathbb{R}^N)$ such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^{p(x)} \, \mathrm{d}x = 0$$

for some $r > 0$, then $u_n \to 0$ in $L^{q(x)}(\mathbb{R}^N)$ for any $q \in C(\mathbb{R}^N, \mathbb{R})$ satisfying $p \ll q \ll p^*$.

Next, we consider the case that V satisfies (V_1). On the linear subspace

$$E = \left\{ u \in W^{1,p(x)}(\mathbb{R}^N) \,\middle|\, \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + V(x) \, |u|^{p(x)} \right) \mathrm{d}x < +\infty \right\},$$

we equip with the norm

$$\|u\|_1 = \inf \left\{ \lambda > 0 \,\middle|\, \int_{\mathbb{R}^N} \left(\left| \frac{\nabla u}{\lambda} \right|^{p(x)} + V(x) \left| \frac{u}{\lambda} \right|^{p(x)} \right) \mathrm{d}x \leq 1 \right\}.$$

Then $(E, \|\cdot\|_1)$ is continuously embedded into $W^{1,p(x)}(\mathbb{R}^N)$ as a closed subspace. Therefore, $(E, \|\cdot\|_1)$ is also a separable reflexive Banach space. It is easy to see that with the norm $\|\cdot\|_1$, Proposition 2.1 remains valid.

Lemma 2.6. *If V satisfies (V_1), then*

(i) *we have a compact embedding $E \hookrightarrow L^{p(x)}(\mathbb{R}^N)$*
(ii) *for any measurable function $q : \mathbb{R}^N \to \mathbb{R}$ with $p < q \ll p^*$, we have a compact embedding $E \hookrightarrow L^{q(x)}(\mathbb{R}^N)$.*

Remark 2.7. The case $p(x) \equiv 2$ is due to Bartsch and Wang [6]. We will adapt their idea to prove the above lemma. For the case that V is coercive, a similar result involving variable exponent can be found in Alves [2]

Proof of Lemma 2.6. (i) Assume $u_n \rightharpoonup \mathbf{0}$ in E. We will show that $u_n \to \mathbf{0}$ in $L^{p(x)}(\mathbb{R}^N)$. For this purpose, since by Proposition 2.4 (ii) we have a compact embedding $E \hookrightarrow L^{p(x)}_{\mathrm{loc}}(\mathbb{R}^N)$, we only need to show that for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{|x|\geq R} |u_n|^{p(x)} \, \mathrm{d}x \leq \varepsilon, \qquad \text{for all } n = 1, 2, \cdots.$$

Note that $\sup_n \|u_n\|_1 < +\infty$. Given $\varepsilon > 0$, set

$$M = \frac{2}{\varepsilon} \sup_n \left(\|u_n\|_1^{p+} + \|u_n\|_1^{p-} \right).$$

Choose a number $s \in \left(1, \frac{N}{N-p_-}\right)$ arbitrarily, then $p(x) < sp(x) < p^*(x)$. By Proposition 2.4, there is a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^N} |u_n|^{sp(x)}\,\mathrm{d}x\right)^{1/s} \le C. \tag{2.5}$$

By assumption (V_1), we may choose R large enough such that

$$\mu\left(\left\{x \in \mathbb{R}^N \,\middle|\, |x| > R, V(x) < M\right\}\right) \le \left(\frac{\varepsilon}{2C}\right)^{s/(s-1)}.$$

Denote

$$A = \left\{x \in \mathbb{R}^N \,\middle|\, |x| \ge R, V(x) \ge M\right\}, \qquad B = \left\{x \in \mathbb{R}^N \,\middle|\, |x| \ge R, V(x) < M\right\}.$$

Then using the $\|\cdot\|_1$ version of Proposition 2.1, we have

$$\begin{aligned}\int_A |u_n|^{p(x)}\,\mathrm{d}x &\le \int_A \frac{V(x)}{M}|u_n|^{p(x)}\,\mathrm{d}x \\ &\le \frac{1}{M}\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)} + V(x)\,|u_n|^{p(x)}\right)\mathrm{d}x \\ &\le \frac{1}{M}\left(\|u_n\|_1^{p_+} + \|u_n\|_1^{p_-}\right) \le \frac{\varepsilon}{2}.\end{aligned} \tag{2.6}$$

On the other hand, by Hölder inequality and (2.5), we have

$$\begin{aligned}\int_B |u_n|^{p(x)}\,\mathrm{d}x &\le \left(\int_B |u_n|^{sp(x)}\,\mathrm{d}x\right)^{1/s}\left(\int_B \mathrm{d}x\right)^{(s-1)/s} \\ &\le C\cdot[\mu(B)]^{(s-1)/s} \le \frac{\varepsilon}{2}.\end{aligned} \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\int_{|x|\ge R} |u_n|^{p(x)}\,\mathrm{d}x = \left(\int_A + \int_B\right)|u_n|^{p(x)}\,\mathrm{d}x \le \varepsilon,$$

from which conclusion (i) of the lemma follows.

(ii) To prove the lemma for general exponent q, we use an interpolation argument. Let $u_n \rightharpoonup \mathbf{0}$ in E, we have just proved that $u_n \to \mathbf{0}$ in $L^{p(x)}(\mathbb{R}^N)$. That is

$$\int_{\mathbb{R}^N} |u_n|^{p(x)}\,\mathrm{d}x \to 0. \tag{2.8}$$

Moreover, because the embedding $E \hookrightarrow L^{p^*(x)}(\mathbb{R}^N)$ is continuous and $\{u_n\}$ is bounded in E, we also have

$$\sup_n \int_{\mathbb{R}^N} |u_n|^{p^*(x)}\mathrm{d}x < \infty. \tag{2.9}$$

Since $p < q \ll p^*$, there exists $\lambda : \mathbb{R}^N \to (0,1)$ such that

$$\frac{1}{q(x)} = \frac{\lambda(x)}{p(x)} + \frac{1-\lambda(x)}{p^*(x)}, \qquad \text{a.e. } x \in \mathbb{R}^N.$$

Then, for $x \in \mathbb{R}^N$ we have

$$s(x) := \frac{p(x)}{q(x)\lambda(x)} > 1, \qquad t(x) := \frac{p^*(x)}{q(x)(1-\lambda(x))} > 1.$$

Using Proposition 2.3 and (2.3), as well as (2.8) and (2.9), we deduce

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n|^{q(x)}\,\mathrm{d}x &= \int_{\mathbb{R}^N} |u_n|^{p(x)/s(x)}\,|u_n|^{p^*(x)/t(x)}\,\mathrm{d}x \le 2\left||u_n|^{p/s}\right|_{s(x)}\left||u_n|^{p^*/t}\right|_{t(x)}\\ &\le 2\left(\left(\int_{\mathbb{R}^N}|u_n|^{p(x)}\,\mathrm{d}x\right)^{1/s_+}+\left(\int_{\mathbb{R}^N}|u_n|^{p(x)}\,\mathrm{d}x\right)^{1/s_-}\right)\\ &\quad\times\left(\left(\int_{\mathbb{R}^N}|u_n|^{p^*(x)}\,\mathrm{d}x\right)^{1/t_+}+\left(\int_{\mathbb{R}^N}|u_n|^{p^*(x)}\,\mathrm{d}x\right)^{1/t_-}\right)\to 0.\end{aligned}$$

This implies $u_n \to \mathbf{0}$ in $L^{q(x)}(\mathbb{R}^N)$, and the proof of conclusion (ii) is complete. □

3. Boundedness of Cerami sequences

For simplicity, from now on we denote $X = W^{1,p(x)}(\mathbb{R}^N)$. Under our assumptions, it is known that $\Phi \in C^1(X,\mathbb{R})$, with derivative given by (1.6). We recall that $u \in X$ is a (weak) solution of (1.1), if for any $\phi \in C_0^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)-2}\,\nabla u\cdot\nabla\phi + V(x)\,|u|^{p(x)-2}\,u\phi\right)\mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)\phi\,\mathrm{d}x = 0. \tag{3.1}$$

Since $C_0^\infty(\mathbb{R}^N) \subset X$, compare (3.1) with (1.6) we see that critical points of $\Phi \in C^1(X,\mathbb{R})$ are weak solutions of (1.1). Thus to find solutions of (1.1) we only need to find critical points of $\Phi \in C^1(X,\mathbb{R})$. Similarly, if we consider Φ as a C^1-functional on E, then critical points of $\Phi \in C^1(E,\mathbb{R})$ are also solutions of (1.1).

Recall that for $c \in \mathbb{R}$, a sequence $\{u_n\} \subset X$ is called a Cerami sequence of Φ at the level c, a $(C)_c$ sequence for short, if

$$\Phi(u_n) \to c, \qquad (1+\|u_n\|)\Phi'(u_n) \to 0.$$

Note that under our assumptions we have

$$F(x,t) \ge 0, \qquad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.2}$$

see [19, eq (2.6)] for the details.

Lemma 3.1. *Suppose that (p), (V), (f_1), (f_2) and (f_3) hold, p, V are periodic, and $f(\cdot,t)$ is periodic for all $t \in \mathbb{R}$. If $c \in \mathbb{R}$, then any $(C)_c$ sequence of Φ is bounded.*

Proof. Assume that Φ has an unbounded $(C)_c$ sequence, $\{u_n\}$. Up to a subsequence we may assume that

$$\Phi(u_n) \to c, \qquad \|u_n\| \to \infty, \qquad \langle \Phi'(u_n), u_n\rangle \to 0.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^N}\left(\frac{1}{p(x)}-\frac{1}{p_+}\right)\left(|\nabla u|^{p(x)}+V(x)\,|u|^{p(x)}\right)\mathrm{d}x + \frac{1}{p_+}\int_{\mathbb{R}^N}\mathcal{F}(x,u_n)\,\mathrm{d}x\\ = \Phi(u_n) - \frac{1}{p_+}\langle\Phi'(u_n),u_n\rangle \to c.\end{aligned} \tag{3.3}$$

Let $v_n = \|u_n\|^{-1}\,u_n$, then $\{v_n\}$ is bounded in X. We claim that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_2(y)}|v_n|^{p(x)}\,\mathrm{d}x = 0. \tag{3.4}$$

Otherwise, up to a subsequence we can choose $\delta > 0$ and $\{z_n\} \subset \mathsf{R}^N$ such that

$$\int_{B_2(z_n)} |v_n|^{p(x)}\,\mathrm{d}x \geq \frac{\delta}{2}.$$

Since the number of points in $\mathsf{Z}^N \cap B_2(z_n)$ is less than 4^N, there exists $y_n \in \mathsf{Z}^N \cap B_2(z_n)$ such that

$$\int_{B_2(y_n)} |v_n|^{p(x)}\,\mathrm{d}x \geq \kappa := \frac{\delta}{2\times 4^N} > 0.$$

Let $\tilde{v}_n = v_n(\cdot + y_n)$, then $\{\tilde{v}_n\}$ is bounded in X. Passing to a subsequence we have

$$\tilde{v}_n \to \tilde{v} \qquad \text{in } L^{p(x)}_{\text{loc}}(\mathsf{R}^N), \qquad \tilde{v}_n(x) \to \tilde{v}(x) \qquad \text{a.e. } x \in \mathsf{R}^N.$$

Since

$$\int_{B_2(0)} |\tilde{v}_n|^{p(x)}\,\mathrm{d}x = \int_{B_2(y_n)} |v_n|^{p(x)}\,\mathrm{d}x \geq \kappa > 0,$$

we see that $\tilde{v} \neq \mathbf{0}$. Let $\tilde{u}_n = \|u_n\|\,\tilde{v}_n$. If $\tilde{v}(x) \neq 0$ we have $|\tilde{u}_n(x)| \to +\infty$. Using (1.3) we obtain

$$\frac{F(x,\tilde{u}_n(x))}{|\tilde{u}_n(x)|^{p+}}\,|\tilde{v}_n(x)|^{p+} \to +\infty. \tag{3.5}$$

Note that $F(x,t)$ is periodic with respect to x, we have

$$\int_{\mathsf{R}^N} F(x,u_n)\,\mathrm{d}x = \int_{\mathsf{R}^N} F(x,\tilde{u}_n)\,\mathrm{d}x.$$

Since the set $\Theta = \left\{x \in \mathsf{R}^N \middle|\ \tilde{v}(x) \neq 0\right\}$ has positive Lebesgue measure and $\|u_n\| > 1$ for n large, using (3.2), (3.5), Proposition 2.1 and the Fatou Lemma we have

$$\begin{aligned}
\frac{1}{p_-} &\geq \frac{1}{p_-\,\|u_n\|^{p+}} \int_{\mathsf{R}^N} \left(|\nabla u_n|^{p(x)} + V(x)\,|u_n|^{p(x)}\right)\mathrm{d}x \\
&\geq \frac{1}{\|u_n\|^{p+}} \int_{\mathsf{R}^N} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + V(x)\,|u_n|^{p(x)}\right)\mathrm{d}x \\
&= \frac{1}{\|u_n\|^{p+}} \left(\Phi(u_n) + \int_{\mathsf{R}^N} F(x,u_n)\mathrm{d}x\right) \\
&\geq \frac{1}{\|u_n\|^{p+}} \int_{\mathsf{R}^N} F(x,u_n)\,\mathrm{d}x - 1 = \int_{\mathsf{R}^N} \frac{F(x,\tilde{u}_n)}{\|u_n\|^{p+}}\mathrm{d}x - 1 \\
&\geq \int_{\tilde{v}\neq 0} \frac{F(x,\tilde{u}_n)}{|\tilde{u}_n|^{p+}}\,|\tilde{v}_n|^{p+}\,\mathrm{d}x - 1 \to +\infty.
\end{aligned} \tag{3.6}$$

This is impossible. Hence (3.4) is true. Applying Proposition 2.5 we see that

$$v_n \to 0 \qquad \text{in } L^{q(x)}(\mathsf{R}^N) \tag{3.7}$$

for the function q from assumption (f_1). We shall derive a contradiction as follow.

Given a real number $r > 1$, by (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x,rt)| \leq \varepsilon\,|t|^{p(x)} + C_\varepsilon\,|t|^{q(x)}. \tag{3.8}$$

Since $\|v_n\| = 1$, by Proposition 2.4, there exists a constant $C_1 > 0$ such that

$$\int_{\mathsf{R}^N} |v_n|^{p(x)}\,\mathrm{d}x \leq C_1.$$

Hence, by (3.7), (3.8), and Remark 2.2, we deduce

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} |F(x, rv_n)| \,\mathrm{d}x \le \overline{\lim_{n\to\infty}} \left\{ \varepsilon \int_{\mathbb{R}^N} |v_n|^{p(x)} \,\mathrm{d}x + C_\varepsilon \int_{\mathbb{R}^N} |v_n|^{q(x)} \,\mathrm{d}x \right\} \le \varepsilon C_1.$$

Now let $\varepsilon \to 0$, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, rv_n) \,\mathrm{d}x = 0. \tag{3.9}$$

As in [15, 28], we choose a sequence $\{t_n\} \subset [0, 1]$ such that

$$\Phi(t_n u_n) = \max_{t\in[0,1]} \Phi(tu_n).$$

For any $m > 1$, we have $r = (2mp_+)^{1/p_-} > 1$. For n large we have $\|u_n\|^{-1} r \in (0, 1)$. Since $\|v_n\| = 1$, by Proposition 2.1 we see that

$$\int_{\mathbb{R}^N} \left(|\nabla v_n|^{p(x)} + V(x) |v_n|^{p(x)} \right) \mathrm{d}x = 1.$$

Hence, for n large enough, (3.9) gives

$$\begin{aligned}
\Phi(t_n u_n) &\ge \Phi\Big(\frac{r}{\|u_n\|} u_n\Big) = \Phi(rv_n) \\
&= \int_{\mathbb{R}^N} \frac{r^{p(x)}}{p(x)} \left(|\nabla v_n|^{p(x)} + V(x) |v_n|^{p(x)} \right) \mathrm{d}x - \int_{\mathbb{R}^N} F(x, rv_n) \,\mathrm{d}x \\
&\ge \frac{r^{p_-}}{p_+} - \int_{\mathbb{R}^N} F(x, rv_n) \,\mathrm{d}x \ge m.
\end{aligned}$$

That is $\Phi(t_n u_n) \to +\infty$. But $\Phi(\mathbf{0}) = 0$, $\Phi(u_n) \to c$, we see that $t_n \in (0, 1)$ and

$$\langle \Phi'(t_n u_n), t_n u_n \rangle = t_n \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=t_n} \Phi(tu_n) = 0.$$

Now using (3.3) and (f_3) we obtain

$$\begin{aligned}
\frac{1}{\theta} \Phi(t_n u_n) &= \frac{1}{\theta} \left(\Phi(t_n u_n) - \frac{1}{p_+} \langle \Phi'(t_n u_n), t_n u_n \rangle \right) \\
&\le \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{p_+} \right) t_n^{p(x)} \left(|\nabla u_n|^{p(x)} + V(x) |u_n|^{p(x)} \right) \mathrm{d}x + \frac{1}{p_+} \int_{\mathbb{R}^N} \frac{\mathcal{F}(x, t_n u_n)}{\theta} \,\mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{p_+} \right) \left(|\nabla u_n|^{p(x)} + V(x) |u_n|^{p(x)} \right) \mathrm{d}x + \frac{1}{p_+} \int_{\mathbb{R}^N} \mathcal{F}(x, u_n) \,\mathrm{d}x \to c.
\end{aligned}$$

This contradicts with $\Phi(t_n u_n) \to +\infty$. Therefore we have proved that $\{u_n\}$ is bounded. □

4. Proof of Theorem 1.1

To prove Theorem 1.1 we will apply a mountain pass type argument to find nonzero critical point of $\Phi \in C^1(X, \mathbb{R})$. Firstly, we need the following result, which is contained in [9, Lemma 3.1].

Proposition 4.1. *Suppose that (p), (V), (f_1) and (f_2) hold, and $\{u_n\}$ is a sequence in X such that $u_n \rightharpoonup u$ in X, and $\Phi'(u_n) \to 0$. Then $\Phi'(u) = 0$.*

Lemma 4.2. *There exist $r>0$ and $\varepsilon>0$ such that $0<\|u\|\le r$ implies*

$$\Phi(u)\ge\varepsilon\|u\|^{p_+},\qquad\langle\Phi'(u),u\rangle\ge\varepsilon\|u\|^{p_+}. \tag{4.1}$$

Proof. Firstly we have

$$\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)}+\left(V(x)-\frac{V_-}{2}\right)|u|^{p(x)}\right)\mathrm{d}x\ge\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)}+V(x)|u|^{p(x)}\right)\mathrm{d}x. \tag{4.2}$$

By (f_1) and (f_2), there exists $C>0$ such that

$$|F(x,t)|\le\frac{V_-}{2p_+}|t|^{p(x)}+C|t|^{q(x)},\qquad(x,t)\in\mathbb{R}^N\times\mathbb{R}. \tag{4.3}$$

Therefore, using (4.2), if $\|u\|\le 1$ we have

$$\begin{aligned}\Phi(u)&\ge\frac{1}{p_+}\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)}+V(x)|u|^{p(x)}\right)\mathrm{d}x-\frac{V_-}{2p_+}\int_{\mathbb{R}^N}|u|^{p(x)}\,\mathrm{d}x-C\int_{\mathbb{R}^N}|u|^{q(x)}\,\mathrm{d}x\\&\ge\frac{1}{2p_+}\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)}+V(x)|u|^{p(x)}\right)\mathrm{d}x-C\int_{\mathbb{R}^N}|u|^{q(x)}\,\mathrm{d}x\\&\ge\frac{1}{2p_+}\|u\|^{p_+}-C_1\|u\|^{q_-}.\end{aligned}$$

Since $p_+\ll q$ implies $p_+<q_-$, the result for $\Phi(u)$ follows. The desired result for $\langle\Phi'(u),u\rangle$ follows similarly. □

Using (1.3), it is easy to see that for any $u\ne\mathbf{0}$, we have $\Phi(tu)\to-\infty$. The following lemma is a consequence of this fact and Lemma 4.2.

Lemma 4.3. *There exist $r>0$ and $e\in X$ such that $\|e\|>r$ and*

$$b:=\inf_{\|u\|=r}\Phi(u)>\Phi(\mathbf{0})=0\ge\Phi(e). \tag{4.4}$$

Proof of Theorem 1.1. By Lemma 4.3 we see that Φ has a mountain pass geometry. That is,

$$\Gamma=\{\gamma\in C([0,1],X)|\ \gamma(0)=\mathbf{0}\text{ and }\Phi(\gamma(0))<0\}\ne\emptyset.$$

By a special version of the Mountain Pass Lemma (see [8]), for the mountain pass level

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi(\gamma(t))>0, \tag{4.5}$$

there exists a $(C)_c$ sequence $\{u_n\}$ for Φ, which is bounded in X via Lemma 3.1.

Let

$$\delta=\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_2(y)}|u_n|^{p(x)}\,\mathrm{d}x.$$

If $\delta=0$, using Proposition 2.5, similar to (3.9) we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}F(x,u_n)\,\mathrm{d}x=0,\qquad\lim_{n\to\infty}\int_{\mathbb{R}^N}f(x,u_n)u_n\,\mathrm{d}x=0. \tag{4.6}$$

Since $\langle\Phi'(u_n),u_n\rangle\to 0$ and

$$\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)}+V(x)|u_n|^{p(x)}\right)=\langle\Phi'(u_n),u_n\rangle+\int_{\mathbb{R}^N}f(x,u_n)u_n\,\mathrm{d}x.$$

we deduce that

$$\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p(x)}+V(x)|u_n|^{p(x)}\right)\to 0.$$

By Proposition 2.1, $\|u_n\| \to 0$ and consequently $\Phi(u_n) \to 0$, a contradiction with $\Phi(u_n) \to c > 0$.

Hence $\delta > 0$. As in the proof of Lemma 3.1, we can choose a sequence $\{y_n\} \subset \mathbb{Z}^N$ such that setting $v_n = u_n(\cdot + y_n)$, we have $v_n \rightharpoonup v \neq \mathbf{0}$ in X. By the $\mathbb{Z}^N$ invariance of the problem, $\{v_n\}$ is also a $(C)_c$ sequence of Φ. Using Proposition 4.1, we see that $\Phi'(v) = 0$. Namely, v is a nontrivial solution of (1.1).

Now we will follow the approach of [17] to show that (1.1) has a ground state. Let us denote

$$\mathcal{C} = \left\{ u \in X \setminus \{\mathbf{0}\} \middle|\ \Phi'(u) = 0 \right\}, \qquad m = \inf_{u \in \mathcal{C}} \Phi(u).$$

Since (f_3) implies that $\mathcal{F}(x,t) \geq 0$, for any $u \in \mathcal{C}$ we have

$$\begin{aligned}
\Phi(u) &= \Phi(u) - \frac{1}{p_+}\langle \Phi'(u), u\rangle \\
&= \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{p_+}\right)\left(|\nabla u|^{p(x)} + V(x)\,|u|^{p(x)}\right) \mathrm{d}x + \frac{1}{p_+}\int_{\mathbb{R}^N} \mathcal{F}(x,u)\,\mathrm{d}x \\
&\geq \frac{1}{p_+}\int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\,\mathrm{d}x \geq 0.
\end{aligned}$$

Therefore $m \geq 0$. Let $\{u_n\} \subset \mathcal{C}$ be such that $\Phi(u_n) \to m$. Since $\Phi'(u_n) = 0$, by Lemma 4.2 we have

$$\|u_n\| \geq r. \tag{4.7}$$

Obviously $\{u_n\}$ is a $(C)_m$ sequence of Φ, by Lemma 3.1 we see that $\{u_n\}$ is bounded. As before, we denote

$$\delta = \lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_2(y)} |u_n|^{p(x)}\,\mathrm{d}x.$$

If $\delta = 0$, similar to (4.6) we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x,u_n)u_n\,\mathrm{d}x = 0.$$

Thus

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + V(x)\,|u_n|^{p(x)}\right) \mathrm{d}x = \langle \Phi'(u_n), u_n\rangle + \int_{\mathbb{R}^N} f(x,u_n)u_n\,\mathrm{d}x \to 0.$$

By Proposition 2.1, we see that $\|u_n\| \to 0$, a contradiction with (4.7).

Therefore, $\delta > 0$. By the $\mathbb{Z}^N$ invariance of Φ, a suitable translation of $\{u_n\}$, denoted by $\{v_n\}$ satisfies

$$\langle \Phi'(v_n), v_n\rangle \to 0, \qquad \Phi(v_n) \to m,$$

and $\{v_n\}$ converges weakly to some $v \neq \mathbf{0}$, a nonzero critical point of Φ. By the Fatou's Lemma we have

$$\int_{\mathbb{R}^N} \mathcal{F}(x,v)\,\mathrm{d}x \leq \varliminf_{n\to\infty} \int_{\mathbb{R}^N} \mathcal{F}(x,v_n)\,\mathrm{d}x. \tag{4.8}$$

Note that $p_- > 1$, it is easy to see that the continuous functional $\varphi : X \to \mathbb{R}$,

$$\varphi(u) = \int_{\mathbb{R}^N} \left(\frac{1}{p(x)} - \frac{1}{p_+}\right)\left(|\nabla u|^{p(x)} + V(x)\,|u|^{p(x)}\right) \mathrm{d}x,$$

is convex. Using [22, Theorem 1.2], we know that φ is weakly lower semi-continuous. Thus

$$\varphi(v) \leq \varliminf_{n\to\infty} \varphi(v_n). \tag{4.9}$$

By (4.8) and (4.9), we have

$$\begin{aligned}\Phi(v) &= \Phi(v) - \frac{1}{p_+}\langle \Phi'(v), v\rangle = \varphi(v) + \frac{1}{p_+}\int_{\mathbb{R}^N} \mathcal{F}(x,v)\,\mathrm{d}x \\ &\le \varliminf_{n\to\infty} \left(\varphi(v_n) + \frac{1}{p_+}\int_{\mathbb{R}^N} \mathcal{F}(x,v)\,\mathrm{d}x\right) \\ &= \varliminf_{n\to\infty} \Phi(v_n) = m.\end{aligned}$$

Therefore, v is a nonzero critical point of Φ with $\Phi(v) = m$. Theorem 1.1 is proved. □

5. Proof of Theorem 1.3

In our proof of Theorem 1.3 we will consider Φ as a functional on $(E, \|\cdot\|_1)$. We say that an operator $L : E \to E^*$ is of (S_+) type if $u_n \rightharpoonup u$ and

$$\varlimsup_{n\to\infty} \langle L(u_n) - L(u), u_n - u\rangle \le 0$$

imply $u_n \to u$ in E.

Lemma 5.1. *Suppose that (p), (V_1), (f_1), (f_2) and (f_3) hold, then Φ satisfies the Cerami condition (C). Namely, for all $c \in \mathbb{R}$, any $(C)_c$ sequence of Φ has a convergent subsequence.*

Proof. Let $\{u_n\} \subset E$ be a $(C)_c$ sequence of Φ. According to Lemma 2.6, the embeddings $E \hookrightarrow L^{p(x)}(\mathbb{R}^N)$ and $E \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ are compact. With this in mind, modifying the proof of Lemma 3.1 slightly, we deduce that $\{u_n\}$ is bounded in E. Up to a subsequence we may assume that $u_n \rightharpoonup u$ in E and

$$u_n \to u \qquad \text{in } L^{q(x)}(\mathbb{R}^N). \tag{5.1}$$

By the boundedness of $\{u_n\}$ in $L^{p(x)}(\mathbb{R}^N)$, we have

$$\Lambda_1 = \sup_n \int_{\mathbb{R}^N} |u_n|^{p(x)}\,\mathrm{d}x < \infty. \tag{5.2}$$

Using (2.3) and the Hölder inequality (2.4) we also have

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n|^{p(x)-1} u\mathrm{d}x &\le 2\left||u_n|^{p(x)-1}\right|_{p'(x)} |u|_{p(x)} \\ &\le 2|u|_{p(x)}\left(\left(\int_{\mathbb{R}^N}\left||u_n|^{p(x)-1}\right|^{p'(x)}\mathrm{d}x\right)^{p'_+} + \left(\int_{\mathbb{R}^N}\left||u_n|^{p(x)-1}\right|^{p'(x)}\mathrm{d}x\right)^{p'_-}\right) \\ &= 2|u|_{p(x)}\left(\left(\int_{\mathbb{R}^N}|u_n|^{p(x)}\mathrm{d}x\right)^{p'_+} + \left(\int_{\mathbb{R}^N}|u_n|^{p(x)}\mathrm{d}x\right)^{p'_-}\right).\end{aligned}$$

Applying (5.2), we deduce

$$\Lambda_3 = \sup_n \int_{\mathbb{R}^N} |u_n|^{p(x)-1} u\mathrm{d}x < \infty.$$

Similarly,

$$\Lambda_4 = \sup_n \int_{\mathbb{R}^N} |u|^{p(x)-1} u_n\mathrm{d}x < \infty.$$

Now, we adapt the argument in the proof of [26, Lemma 3.11]. For any $\varepsilon > 0$, choose $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon |t|^{p(x)-1} + C_\varepsilon |t|^{q(x)-1}, \qquad \text{all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Then using (2.4) again we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u)\,\mathrm{d}x \\
&\quad \le \int_{\mathbb{R}^N} \left[\varepsilon\left(|u_n|^{p(x)-1} + |u|^{p(x)-1}\right) + C_\varepsilon\left(|u_n|^{q(x)-1} + |u|^{q(x)-1}\right)\right] |u_n - u|\,\mathrm{d}x \\
&\quad \le \varepsilon \int_{\mathbb{R}^N} \left(|u_n|^{p(x)} + |u|^{p(x)} + |u_n|^{p(x)-1}|u| + |u|^{p(x)-1}|u_n|\right)\mathrm{d}x \\
&\qquad + C_\varepsilon\left(\int_{\mathbb{R}^N} |u_n|^{q(x)-1}|u_n - u|\,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{q(x)-1}|u_n - u|\,\mathrm{d}x\right) \\
&\quad \le \left(\Lambda_1 + \Lambda_3 + \Lambda_4 + \int_{\mathbb{R}^N} |u|^{p(x)}\,\mathrm{d}x\right)\varepsilon \\
&\qquad + 2C_\varepsilon\left(\sup_n \left||u_n|^{q(x)-1}\right|_{q'(x)} + \left||u|^{q(x)-1}\right|_{q'(x)}\right)|u_n - u|_{q(x)}.
\end{aligned} \tag{5.3}$$

Since $\{u_n\}$ is bounded in $L^{q(x)}(\mathbb{R}^N)$, it follows that $\left\{|u_n|^{q(x)-1}\right\}$ is bounded in $L^{q'(x)}(\mathbb{R}^N)$. That is

$$\sup_n \left||u_n|^{q(x)-1}\right|_{q'(x)} < \infty.$$

Therefore we can deduce from (5.1) and (5.3) that

$$\int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u)\,\mathrm{d}x \to 0.$$

Note that $\Phi'(u_n) \to 0$, we have

$$\begin{aligned}
\langle \psi'(u_n) - \psi'(u), u_n - u\rangle &= \langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \\
&\quad + \int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))(u_n - u)\,\mathrm{d}x \to 0.
\end{aligned}$$

Since ψ' is of (S_+) type ([12, Theorem 3.1]), we obtain $u_n \to u$ in E. The proof is complete. □

To prove Theorem 1.3 we need the Fountain Theorem of Bartsch [5, Theorem 2.5], see also [26, Theorem 3.6].

Let X be a reflexive and separable Banach space. It is well known that there exist $\{v_n\}_{n\in\mathbb{N}} \subset X$ and $\{\varphi_n\}_{n\in\mathbb{N}} \subset X^*$ such that

(i) $\langle \varphi_n, v_m\rangle = \delta_{n,m}$, where $\delta_{n,m} = 1$ for $n = m$ and $\delta_{n,m} = 0$ for $n \ne m$.
(ii) $\overline{\operatorname{span}}\{v_n;\, n \in \mathbb{N}\} = X$, $\overline{\operatorname{span}}^{w^*}\{\varphi_n;\, n \in \mathbb{N}\} = X^*$.

Let $X_j = \mathbb{R}v_j$, then $X = \overline{\bigoplus_{j\ge1} X_j}$. Now we define

$$Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j\ge k} X_j}. \tag{5.4}$$

Then we have the following Fountain Theorem.

Theorem 5.2 (Fountain Theorem)**.** *Assume that $\Phi \in C^1(X,\mathbb{R})$ satisfies the Cerami condition (C), $\Phi(-u) = \Phi(u)$. If for almost every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(i) $b_k = \inf_{u \in Z_k, \|u\| = r_k} \Phi(u) \to +\infty$, *as* $k \to \infty$,

(ii) $a_k = \max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \leq 0$,

then Φ *has a sequence of critical points* $\{u_k\}$ *such that* $\Phi(u_k) \to +\infty$.

Remark 5.3. In [5, 26], the Fountain Theorem is proved assuming that Φ satisfies the Palais-Smale (PS) condition. Since the Deformation Theorem is still valid under the Cerami condition, we see that like many critical point theorems, the Fountain Theorem is true under the Cerami condition.

Proof of Theorem 1.3. Since Φ satisfies the Cerami condition (C) and has the mountain pass geometry, using the Mountain Pass Lemma, the conclusion (i) of the theorem follows readily.

To prove conclusion (ii), for the reflexive and separable Banach space E, define Y_k and Z_k as in (5.4). We know that Φ satisfies the condition (C), and $\Phi(-u) = \Phi(u)$. So to prove Theorem 1.3 (ii), it remains to verify the conditions (i) and (ii) of Theorem 5.2.

Verification of (i). Consider the functional $\Psi : E \to \mathsf{R}$,

$$\Psi(u) = \int_{\mathsf{R}^N} |u|^{q(x)} \, dx.$$

Since the embedding $E \hookrightarrow L^{q(x)}(\mathsf{R}^N)$ is compact, it is easy to show that Ψ is weakly continuous, namely $u_n \rightharpoonup u$ implies $\Psi(u_n) \to \Psi(u)$. Hence using [10, Lemma 3.3] we have

$$\beta_k := \sup_{u \in Z_k, \|u\|_1 = 1} |\Psi(u)| \to 0, \qquad \text{as } k \to \infty.$$

By the above definition of β_k, for $u \in Z_k$ with $\|u\|_1 \geq 1$, we have

$$\int_{\mathsf{R}^N} |u|^{q(x)} \, dx \leq \beta_k \, \|u\|_1^{q_+} . \tag{5.5}$$

Choose a constant $C > 0$ satisfying (4.3), then consider the real function $\eta : \mathsf{R} \to \mathsf{R}$,

$$\eta(r) = \frac{1}{2p_+} r^{p_-} - C\beta_k r^{q_+}.$$

Using elementary calculus, it is easy to see that η attains its maximum value at

$$r_k = \left(\frac{2Cp_+q_+\beta_k}{p_-} \right)^{1/(p_- - q_+)}.$$

The maximum value

$$\begin{aligned}
\eta(r_k) &= \frac{1}{2p_+} \left[\left(\frac{2Cp_+q_+\beta_k}{p_-} \right)^{p_-/(p_- - q_+)} - 2p_+C\beta_k \left(\frac{2Cp_+q_+\beta_k}{p_-} \right)^{q_+/(p_- - q_+)} \right] \\
&= \frac{1}{2p_+} \left(2p_+C\beta_k \right)^{p_-/(p_- - q_+)} \left[\left(\frac{q_+}{p_-} \right)^{p_-/(p_- - q_+)} - \left(\frac{q_+}{p_-} \right)^{q_+/(p_- - q_+)} \right] \\
&= \frac{1}{2p_+} \left(2p_+C\beta_k \right)^{p_-/(p_- - q_+)} \left(\frac{q_+}{p_-} \right)^{p_-/(p_- - q_+)} \left(1 - \frac{p_-}{q_+} \right).
\end{aligned}$$

Since $p_- < q_+$ and $\beta_k \to 0$, we see that

$$\eta(r_k) \to +\infty, \qquad \text{as } k \to \infty. \tag{5.6}$$

We also have $r_k \to \infty$. For $u \in Z_k$, $\|u\|_1 = r_k$, using (4.3), (4.2) and (5.5) we deduce

$$\begin{aligned}\Phi(u) &\geq \frac{1}{2p_+}\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)} + V(x)\,|u|^{p(x)}\right)\mathrm{d}x - C\int_{\mathbb{R}^N}|u|^{q(x)}\,\mathrm{d}x\\ &\geq \frac{1}{2p_+}\,\|u\|_1^{p_-} - C\beta_k\,\|u\|_1^{q_+} = \eta(r_k)\end{aligned}$$

Using (5.6), it readily follows that

$$b_k = \inf_{u\in Z_k,\|u\|_1=r_k}\Phi(u) \to +\infty.$$

Verification of (ii). The most simple method for this purpose is the following indirect argument, which does not require that $F(x,t)$ grows faster than $|t|^{\mu}$ for some $\mu > p_+$.

Assume that (ii) of Theorem 5.2 does not hold for some given k. Then there exists a sequence $\{u_n\} \subset Y_k$ such that

$$\|u_n\|_1 \to \infty, \qquad \Phi(u_n) \geq 0. \tag{5.7}$$

Let $v_n = \|u_n\|_1^{-1}\,u_n$, then $\|v_n\|_1 = 1$. Since $\dim Y_k < \infty$, there exists $v \in Y_k\setminus\{\mathbf{0}\}$ such that up to a subsequence,

$$\|v_n - v\|_1 \to 0, \qquad v_n(x) \to v(x) \quad \text{a.e. } x \in \mathbb{R}^N.$$

If $v(x) \neq 0$ then $|u_n(x)| \to \infty$. Consequently, by (1.3) we have

$$\frac{F(x,u_n(x))}{|u_n(x)|^{p_+}}\,|v_n(x)|^{p_+} \to +\infty. \tag{5.8}$$

Similar to (3.6), using (3.2), (5.8) and applying the Fatou Lemma, we obtain

$$\int_{\mathbb{R}^N}\frac{F(x,u_n)}{\|u_n\|_1^{p_+}}\mathrm{d}x = \int_{\mathbb{R}^N}\frac{F(x,u_n)}{|u_n|^{p_+}}\,|v_n|^{p_+}\,\mathrm{d}x \to +\infty.$$

We may assume that $\|u_n\|_1 \geq 1$, so

$$\begin{aligned}\Phi(u_n) &\leq \frac{1}{p_-}\,\|u_n\|_1^{p_+} - \int_{\mathbb{R}^N}F(x,u_n)\,\mathrm{d}x\\ &= \frac{1}{p_-}\,\|u_n\|_1^{p_+}\left(1 - \int_{\mathbb{R}^N}\frac{F(x,u_n)}{\|u_n\|_1^{p_+}}\mathrm{d}x\right) \to -\infty,\end{aligned}$$

a contradiction with (5.7). □

6. Some remarks on bounded domain problems

The arguments used in this paper can also be applied to study the existence and multiplicity of solutions for some problems involving the $p(x)$-Laplacian operator in bounded domain. More precisely, we can consider quasilinear problems like

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega.\end{cases} \tag{6.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and f is a function verifying:

(f_*) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $|f(x,t)| \leq C_1(1 + |t|^{q(x)})$ for some $p \ll q \ll p^*$.

(f_1^*) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{p+}} = +\infty.$

(f_2^*) There exist $\theta \geq 1$ and $C_2 > 0$ such that $\theta\mathcal{F}(x,t) \geq \mathcal{F}(x,st) - C_2$, $s \in [0,1]$. where $\mathcal{F}(x,t) = f(x,t)t - p_+F(x,t)$.

Note that according to [20, Lemma 2.3], (f_2^*) is weaker than

(f_2') there exists $R > 0$ such that $\frac{f(x,t)}{|t|^{p+-1}}$ is increasing in t for $|t| \geq R$.

Therefore the condition (f_2^*) is rather weak. Our main result is the following theorem.

Theorem 6.1. *Assume* (f_*), (f_1^*) *and* (f_2^*).

(i) *If* $f(x,t) = o(|t|^{p(x)-1})$ *as* $t \to 0$, *then* (6.1) *has a nontrivial solution.*

(ii) *If* $f(x,t)$ *is odd in* t *and there exists* $\Lambda > 0$ *such that*

$$F(x,t) \geq -\Lambda\,|t|^{p+}, \qquad (x,t) \in \Omega \times \mathsf{R} \tag{6.2}$$

then (6.1) *has a sequence of solutions* $\{u_n\}$ *such that* $\Phi(u_n) \to +\infty$.

Remark 6.2. (i) Because of (f_1^*), actually (6.2) is a local assumption near $t = 0$.

(ii) The case that $p(x)$ is a constant $p \in (1, N)$ has been considered in Liu [20].

(iii) The case that $f(x,t)$ satisfies Ambrosetti-Rabinowitz type condition (AR) has been studied by Fan and Zhang [12, Theorems 4.7 and 4.8]

(iv) To removed (AR), Zang [27] required the condition $(\mathcal{F})$ mentioned in Section 1. Obviously the condition $(\mathcal{F})$ is much stronger than our condition (f_2^*).

To prove Theorem 6.1, we denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\mathsf{R}^N)$. As before, it suffices to find critical points of the C^1-functional $\Phi : W_0^{1,p(x)}(\Omega) \to \mathsf{R}$,

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x.$$

Since Ω is bounded, as a consequence of Proposition 2.4 (ii) we have a compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$. Using this compact embedding, we may adapt the argument in the proof of Lemmas 3.1 and 5.1 to show that Φ satisfies the Cerami condition (C). Note that since (3.2) is no longer true under our weaker condition (f_2^*), to obtain a conclusion similar to (3.6) we need (6.2). Then the Mountain Pass Lemma and the Fountain Theorem can be employed to prove Theorem 6.1 (i) and (ii), respectively. We omit the details.

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