Lyapunov-Schmidt reduction, critical groups and mulltiple solutions of elliptic resonant problems*

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1. Morse theory

Let X be a Hilbert space, $f \in C^1(X, \mathrm{R})$ satisfies (PS). If u is an isolated critical point of f, $f(u) = c$, then

$$C_q(f, u) := H_q(f_c, f_c \setminus \{u\}), \qquad q = 0, 1, \cdots;$$

$f_c = f^{-1}(-\infty, c]$. If the critical values of f are greater than α, then

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We have the Morse inequalities

$$M_q - M_1 + \cdots + (-1)^q M_0 \geq \beta_q - \beta_1 + \cdots + (-1)^q \beta_0, \qquad \forall q \in \mathrm{N}$$

$$M_0 - M_1 + \cdots + (-1)^q M_q + \cdots = \beta_0 - \beta_1 + \cdots + (-1)^q \beta_q + \cdots, \qquad (1)$$

where $\qquad M_q = \sum_{f'(u)=0} \mathrm{rank} C_q(f,u), \qquad \beta_q = \mathrm{rank} C_q(f,\infty).$

Pro 1. *Suppose $f \in C^1(X, \mathrm{R})$ satisfies (PS). If for some $k \in \mathrm{N}$:*

(i) *$C_k(f, \infty) \neq 0$, then f has critical point u with $C_k(f, u) \neq 0$.*

(ii) *$C_k(f, \infty) \neq C_k(f, 0)$, then f has a nonzero critical point.*

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Pro 2. *Let $f \in C^1(X, \mathbb{R})$, $X = X^- \oplus X^+$, $k = \dim X^- < \infty$, $\rho > 0$. If*

$$\begin{aligned} f(u) &\le 0, \quad u \in X^-, \ \|u\| \le \rho, \\ f(u) &\ge 0, \quad u \in X^+, \ \|u\| \le \rho, \end{aligned}$$

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Pro 3. *Let $f \in C^1(X, \mathbb{R})$ satisfies (PS), $X = X^- \oplus X^+$, $k = \dim X^- < \infty$. If f is bounded from below on X^+ and*

$$f(u) \to -\infty, \qquad \textit{as } u \in X^-,\ \|u\| \to \infty,$$

then $C_k(f, \infty) \neq 0$.

2. L-S reduction and critical groups

Pro 4. *Let X be a Hilbert space, $X^\pm$ closed subspaces such that $X = X^- \oplus X^+$. If $f \in C^1(X, \mathrm{R})$ and $\kappa > 0$ such that*

$$\pm \langle \nabla f(v + w_1) - \nabla f(v + w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2 \qquad (\mathrm{E}_\pm)$$

for $v \in X^-$ and $w_1, w_2 \in X^+$. Then, there is $\psi : X^- \to X^+$,

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(ii) *If (E_-) holds, then*

$$\varphi(v) := f(v + \psi(v)) = \max_{w \in X^+} f(v + w).$$

Moreover, $\varphi \in C^1(X^-, \mathrm{R})$,
$v \in X^-$ is a critical point of φ iff $v + \psi(v)$ is a critical point of f.

In [Liu-Li, 03], we showed that: Suppose $X = X^- \oplus X^+$,

Thm 5. (i) *If* $\langle \nabla f(v+w_1) - \nabla f(v+w_2), w_1 - w_2 \rangle \geq \kappa \|w_1 - w_2\|^2$,

$$\text{then} \qquad C_q(f,\infty) \cong C_q(\varphi,\infty)\,, \qquad q = 0, 1, \cdots. \tag{2}$$

Moreover, if $k = \dim X^- < \infty$ *and* $C_k(f,\infty) \neq 0$ *, then*

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$$C_0(\varphi,\infty) \neq 0 \Longrightarrow C_q(\varphi,\infty) \cong \delta_{q,0}\mathrm{G},$$
$$C_k(\varphi,\infty) \neq 0 \Longrightarrow C_q(\varphi,\infty) \cong \delta_{q,k}\mathrm{G}.$$

Consider the ellitptic BVP

$$\begin{cases} -\Delta u = p(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Here $p \in C(\Omega \times R, R)$, $$\lim_{|t|\to\infty} \frac{p(x,t)}{t} = \lambda. \tag{4}$$

The solutions of (3) are critical points of $f : X = H_0^1(\Omega) \to R$,

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega P(x,u)\mathrm{d}x \ .$$

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$\lambda = \lambda_\ell$, if for some $\alpha \in [0,1)$,

$$|p(x,t) - \lambda_\ell t| \le C(1+|t|^\alpha), \quad \lim_{|t|\to\infty} \frac{1}{|t|^{2\alpha}}\left(P(x,t) - \frac{\lambda_\ell}{2}t^2\right) = \pm\infty,$$

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then $C_k(f,\infty) \cong \delta_{q,\ell}\mathrm{G}$ for '+',
$C_k(f,\infty) \cong \delta_{q,\ell-1}\mathrm{G}$ for '−', see [Li-Liu, 99].

In what follows we do not require (4).

(p_+) there exist $\varepsilon, R > 0$ such that for $|t| \geq R$,

$$\lambda_\ell \leq \frac{p(x,t)}{t} \leq \lambda_{\ell+1} - \varepsilon, \qquad \lim_{|t|\to\infty} \frac{1}{|t|}\left(P(x,t) - \frac{\lambda_\ell}{2}t^2\right) = +\infty.$$

(p_-) there exist $\varepsilon, R > 0$ such that for $|t| \geq R$,

$$\lambda_{\ell-1} + \varepsilon \leq \frac{p(x,t)}{t} \leq \lambda_\ell, \qquad \lim_{|t|\to\infty} \frac{1}{|t|}\left(P(x,t) - \frac{\lambda_\ell}{2}t^2\right) = -\infty.$$

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Pro 7. (i) *If* (p_+) *holds, then* $C_q(f,\infty) \cong \delta_{q,\ell}\mathrm{G}$, *provided*

$$\frac{p(x,t) - p(x,s)}{t - s} \leq \beta, \qquad \text{for some } \beta < \lambda_{\ell+1}. \tag{5}$$

(ii) *If* (p_-) *holds, then* $C_q(f,\infty) \cong \delta_{q,\ell-1}\mathrm{G}$, *provided*

$$\frac{p(x,t) - p(x,s)}{t - s} \geq \beta, \qquad \text{for some } \beta \geq \lambda_{\ell-1}. \tag{6}$$

The fun f satisfies (PS), Pro 3 and $(\mathrm{E}_\pm)$. Thm 5 is applied.

3. Reduction, homotopy and Alexander duality

Assume that for some $\Lambda > 0$, $|p(x,t) - p(x,s)| \le \Lambda |t - s|$. (7)

Thm 8 (Nonl. Anal., 08). (i) *if (7), (p_-) hold, then* $C_q(f,\infty) \cong \delta_{q,\ell-1}\mathrm{G}$.

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Rek 9. Thm 8 improves [Li-Perera-Su, 01, Pro 1.3]. For $|t| \ge R$,

$$\lambda_{\ell-1} + \varepsilon \le \frac{p(x,t)}{t} \le \lambda_\ell,\ P(x,t) \le \frac{\lambda_\ell - \varepsilon}{2}|t|^2 \Rightarrow C_q(f,\infty) \cong \delta_{q,\ell-1}\mathrm{G};$$

$$\lambda_\ell \le \frac{p(x,t)}{t} \le \lambda_{\ell+1} - \varepsilon,\ P(x,t) \ge \frac{\lambda_\ell + \varepsilon}{2}|t|^2 \Rightarrow C_\ell(f,\infty) \ne 0.$$

Incompatible with $\lim_{|t|\to\infty} \frac{p(x,t)}{t} = \lambda_\ell$.

(p_0) there is $\rho > 0$, $\lambda_k t^2 \le 2P(x,t) \le \lambda_{k+1} t^2$ for $|t| \le \rho$.

Thm 10 (Nonl. Anal., 08). *(3) has a nonzero solution in case*

(i) (p_0), *(7) and* (p_+) *hold,* $k \ne \ell$*; or*

(ii) (p_0), *(7) and* (p_-) *hold,* $k \ne \ell - 1$.

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Pf. By (p_0), f has local linking at 0, so $C_k(f,0) \neq 0$. By Thm 8

$$C_q(f,\infty) \cong \delta_{q,\ell} G \ \text{ in (i)}, \qquad C_q(f,\infty) \cong \delta_{q,\ell-1} G \ \text{ in (ii)}.$$

By Pro 1, the result follows. ■

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For the proof of Thm 8, we use homotopy method.

Pro 11 ([Li-Perera-Su, 01, Thm 3.1]). *If* $\{f_t\} \subset C^{2-0}_{\mathrm{loc}}(X,\mathrm{R})$ *satisfies* (PS), *and there exist* $\alpha \in \mathrm{R}$ *and* $\delta > 0$ *such that*

$$f_t(u) \le \alpha \implies \|f_t'(u)\| \ge \delta, \tag{8}$$

then $(f_0)_\alpha \approx (f_1)_\alpha$. *In particular,* $C_q(f_0,\infty) \cong C_q(f_1,\infty)$ *for all* $q \in \mathrm{N}$.

3.1. Proof of Thm 8 (i)

Define $A_\ell : X \to X$ by

$$\langle A_\ell(u), v\rangle = \int_\Omega (\nabla u \nabla v - \lambda_\ell uv)\mathrm{d}x, \qquad u, v \in X. \tag{9}$$

Let $X^0 = \ker(A_\ell)$, $X^\pm$ be the '±'-space of A_ℓ.

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$$f_t(u) = (1-t)f(u) + t(\|u^+\|^2 + \|u^0\|^2 - \|u^-\|^2), \quad t \in [0,1] \ .$$

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$$f_t(u) = (1-t)f(u) + t(\|u^+\|^2 + \|u^0\|^2 - \|u^-\|^2), \quad t \in [0,1]\ .$$

It is standard to show that f satisfies (PS). Adapt that argument,

$$\left.\begin{array}{r} f'_{t_n}(u_n) \to 0 \\ \|u_n\| \to \infty \end{array}\right\} \Longrightarrow f_{t_n}(u_n) \to +\infty. \tag{10}$$

This implies (8), and Pro 11 yields

$$C_q(f,\infty) = C_q(f_0,\infty) \cong C_q(f_1,\infty) \cong \delta_{q,\dim X^-}\mathrm{G} = \delta_{q,\ell-1}\mathrm{G}. \qquad \blacksquare$$

3.2. Proof of Thm 8 (ii)

Let $h(u) = \frac{1}{2}\langle A_\ell(u), u\rangle - \|u^0\|^2$, and Consider

$$f_t(u) = (1-t)f(u) + t \cdot h(u), \quad t \in [0,1].$$

As in §3.1 we have a claim 'dual' to (10):

$$\left.\begin{array}{r} f'_{t_n}(u_n) \to 0 \\ \|u_n\| \to \infty \end{array}\right\} \Longrightarrow f_{t_n}(u_n) \to -\infty.$$

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However, we can not deduce $C_q(f_0,\infty)\cong C_q(f_1,\infty)$.
By (7) we can apply Thm 5 to reduce the problem into a finite dim space, then apply the Alexander Duality Theorem.

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$$f_t(u) = (1-t)f(u) + t\cdot h(u), \quad t\in[0,1].$$

As in §3.1 we have a claim 'dual' to (10):

$$\left.\begin{array}{r} f'_{t_n}(u_n)\to 0 \\ \|u_n\|\to\infty \end{array}\right\} \Longrightarrow f_{t_n}(u_n)\to -\infty. \tag{11}$$

However, we can not deduce $C_q(f_0,\infty)\cong C_q(f_1,\infty)$.
By (7) we can apply Thm 5 to reduce the problem into a finite dim space, then apply the Alexander Duality Theorem.
Let $\varphi\in C^1(X,\mathrm{R})$ satisfies (PS). If its critical values are greater than α, then

$$\mathfrak{C}^q(\varphi,\infty) := H^q(X,\varphi_\alpha), \qquad q = 0,1,\cdots.$$

Lem 12 (dual). *If $\varphi\in C^1(\mathrm{R}^m,\mathrm{R})$ satisfies (PS), with critical values bounded, then $C_q(\varphi,\infty)\cong\mathfrak{C}^{m-q}(-\varphi,\infty)$.*

Pf. If the critical values of φ are contained in $(-\alpha, \alpha)$, then

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Pull the sets into S^m via the stereographic projection, apply the ADT, then pull back to R^m. The result follows. ■

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Pf of Thm 8 (ii). Since $\lim_{k\to\infty} \lambda_k = +\infty$, there is $m > \ell$,

$$d := \max\{\Lambda, \lambda_\ell\} \le \lambda_m < \lambda_{m+1}.$$

Let Y be the '+' space of A_m, $Z = Y^\perp$. Then $\dim Z = m$. Using $X^0 \subset Z$ and (7), for $z \in Z$ and $y_1, y_2 \in Y$,

$$\langle \nabla f_t(z + y_1) - \nabla f_t(z + y_2), y_1 - y_2 \rangle \ge \kappa \|y_1 - y_2\|^2,$$

where $\kappa = 1 - \lambda_{m+1}^{-1} d > 0$.

Pf. If the critical values of φ are contained in $(-\alpha, \alpha)$, then

$$C_q(\varphi, \infty) = H_q(\mathrm{R}^m, \varphi_{-\alpha}) \cong H_q(\{\varphi < \alpha\}, \{\varphi < -\alpha\}),$$
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where $\kappa = 1 - \lambda_{m+1}^{-1} d > 0$. By Thm 5, there is $\varphi_t \in C^{2-0}_{\mathrm{loc}}(Z, \mathrm{R})$ such that

$$C_q(f_t, \infty) \cong C_q(\varphi_t, \infty).$$

From (11) and the analytic property of the L-S reduction that

$$\left.\begin{array}{r}\varphi'_{t_n}(z_n)\to 0\\ \|z_n\|\to\infty\end{array}\right\}\Longrightarrow(-\varphi)_{t_n}(z_n)\to+\infty.$$

By Pro 11, $(-\varphi_0)_\alpha\approx(-\varphi_1)_\alpha$ and $\mathfrak{C}^q(-\varphi_0,\infty)\cong\mathfrak{C}^q(-\varphi_1,\infty)$.

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By Lem 12,
$$\begin{aligned} C_q(f, \infty) = C_q(f_0, \infty) &\cong C_q(\varphi_0, \infty) \\ &\cong \mathfrak{C}^{m-q}(-\varphi_0, \infty) \cong \mathfrak{C}^{m-q}(-\varphi_1, \infty) \\ &\cong C_q(f_1, \infty) \\ &\cong \delta_{q,\ell} G. \end{aligned}$$

■

4. Multiple solutions of elliptic BVP

Consider

$$\begin{cases} -\Delta u = p(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \qquad \text{let } p_\infty = \lim_{|t|\to\infty} \frac{p(x,t)}{t}. \qquad (12)$$

(p_1) $p \in C^1(\Omega \times \mathrm{R}, \mathrm{R})$, $p(x,0) = 0$, $\dfrac{\partial p(x,0)}{\partial t} < \lambda_1 < p_\infty = \lambda_\ell$.

(p_2) for some $\gamma < \lambda_{\ell+1}$, $\dfrac{\partial p(x,t)}{\partial t} \leq \gamma$. enable to use reduction.

Solutions of (12) are critical points of $f : H_0^1(\Omega) \to \mathrm{R}$,

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega P(x,u)\mathrm{d}x.$$

If p_∞ is not an eigenvalue, [Castro-Cossio, 94] obtained 5 solutions for (12).

For $p_\infty = \lambda_\ell$, assume in addition

- $\star$ $\exists \alpha \in [0,1)$, $c > 0$, such that $|p(x,t) - \lambda_\ell t| \le c(1 + |t|^\alpha)$,

$$\lim_{|t|\to\infty} \frac{1}{|t|^{2\alpha}} \left\{ P(x,t) - \frac{\lambda_\ell}{2} t^2 \right\} = +\infty,$$

[Li-Zhang, 99] also obtained 5 solutions.
In [Liu-Li, 03], we obtained 5 solutions under (p_1), (p_2) and

$$\lim_{|t|\to\infty} \frac{1}{|t|} \left\{ P(x,t) - \frac{\lambda_\ell}{2} t^2 \right\} = +\infty.$$

As in [Li-Zhang, 99], the proof is based on Morse theory, but we used Thm 5 (i) to compute $C_q(f,\infty)$, see Pro 7 .

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Thm 13 (JMAA, 07). *Assume* (p_1), (p_2) *and*

$$\lim_{|t|\to\infty} \left\{ P(x,t) - \frac{\lambda_\ell}{2} t^2 \right\} = +\infty, \text{ then (12) has 5 solutions.}$$

Unlike [Castro-Cossio, 94, Li-Zhang, 99, Liu-Li, 03], now f does not satisfy (PS) . So we can not apply Morse theory to f.

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The Pf is based on the following observation:

(i) Since $(0, p_\infty), (p_\infty, 0) \notin \Sigma_2$, the Fučik spectrum, $f_\pm$ does satisfy (PS). This will produce two critical points $u_\pm$ of mountain pass type,

$$C_q(f, u_\pm) \cong \delta_{q,1} \mathrm{G}.$$

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(ii) The L-S reduced functional

$$\varphi : X^- = \mathrm{span}\,\{\phi_1, \cdots, \phi_\ell\} \to \mathrm{R},$$

is anti-coercive, hence (PS). This was proved by Liu-Tang-Wu [Liu-Tang-Wu, 00], but we give a much simple proof.

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(iii) Eventually, we apply Morse theory to φ.

Lem 14. *If (E_+) in Pro 4 holds, and v is a critical point of φ, then $C_q(\varphi, v) \cong C_q(f, v + \psi(v))$.*

Pf of Thm 13. First note that

$$C_q(f, 0) \cong \delta_{q,0}\mathrm{G}, \qquad C_q(f, u_\pm) \cong \delta_{q,1}\mathrm{G}.$$

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$$C_q(\varphi, \theta) \cong \delta_{q,0}\mathrm{G}\ , \qquad C_q(\varphi, v_\pm) \cong \delta_{q,1}\mathrm{G}.$$

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Since φ is anti-coercive and $\ell = \dim X^- < \infty$, we deduce

(i) φ attains its maximum at $v \in X^-$, so $C_q(\varphi,v) \cong \delta_{q,\ell}\mathrm{G}$.

(ii) Using exact homology sequence, $C_q(\varphi,\infty) \cong \delta_{q,\ell}\mathrm{G}$.

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If φ has no critical points other than θ, v_+, v_- and v, the Morse inequality (1) becomes

$$(-1)^0+2\times(-1)^1+(-1)^\ell=(-1)^\ell,\qquad \text{impossible!}$$

Therefore φ, and hence f, has at least 5 critical points. ■

Next, we consider the case that 0 is not a local min of f.

[Li-Willem, 98] obtained 3 solutions of (12) assume $p \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$, $|p(x,t) - \lambda_\ell t| \le c(1 + |t|^\alpha)$ for some $\alpha \in [0,1)$.

$$\lim_{|t|\to 0} \frac{p(x,t)}{t} = p_0, \qquad \lim_{|t|\to\infty} \frac{p(x,t)}{t} = \lambda_\ell,$$

$$\frac{\partial p(x,t)}{\partial t} \ge \gamma > \lambda_{\ell-1}, \qquad \lim_{|t|\to\infty} \frac{1}{|t|^{2\alpha}} \left\{ P(x,t) - \frac{\lambda_\ell}{2} t^2 \right\} = -\infty.$$

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These conditions imply (PS), and Morse theory is applied.

We improve their result. Assume

(p_*) $p \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $\exists \Lambda > 0$ s.t. $|p(x,t)| \le \Lambda |t|$.

(p_0) $\exists \delta > 0$ and $k \ge \ell$, $\lambda_k t^2 \le 2P(x,t) \le \lambda_{k+1} t^2$ for $|t| \le \delta$.

(p_1) $\exists \gamma > \lambda_{\ell-1}$ such that

$$\frac{p(x,t) - p(x,s)}{t-s} \ge \gamma, \quad \lim_{|t|\to\infty} \left\{ P(x,t) - \frac{\lambda_\ell}{2} t^2 \right\} = -\infty. \quad (13)$$

Thm 15 (P Roy Soc Edinb, 07). *Assume* (p_*), (p_0) *and* (p_1), *then (12) has 3 solutions.*

Unlike [Li-Willem, 98] here the functional $f : H_0^1(\Omega) \to \mathrm{R}$,

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega P(x,u)\mathrm{d}x$$

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By (14), one considers the L-S reduction $\varphi : X^- \to R$. Since $\dim X^- < \infty$, it is easy to verify that φ is anti-coercive thus satisfies (PS).

Our situation is more delicate, because the reduced functional is defined on an infinite dim subspace.

Pf of Thm 15. Let X^+ be the '$-$'-space of A_ℓ and $X^- = (X^+)^\perp$. Then $X = X^- \oplus X^+$, for $v \in X^-$ and $w_1, w_2 \in X^+$, we have

$$-\langle \nabla f(v+w_1) - \nabla f(v+w_2), w_1 - w_2 \rangle \geq \left(\frac{\gamma}{\lambda_{\ell-1}} - 1 \right) \|w_1 - w_2\|^2.$$

Let $\varphi : X^- \to \mathrm{R}$ as described in Pro 4. We want to show

$$\lim_{\|v\| \to \infty} \varphi(v) = +\infty.$$

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Naturally, one picks $\{v_n\} \subset X^-$ such that $\|v_n\| \to \infty$. Then $\left\{ \|v_n\|^{-1} v_n \right\}$ is bounded and converges weakly up to a subsequence. But $\dim X^- = \infty$, the weak limit may be 0. This makes it difficult to show (15).

We consider $f_1 = f|_{X^-}$, then $\nabla f_1 = P_{X^-}\nabla f$, for $v, \phi \in X^-$

$$\langle \nabla f_1(v), \phi \rangle = \langle \nabla f(v), \phi \rangle = \int_\Omega \nabla v \nabla \phi \mathrm{d}x - \int_\Omega p(x, v)\phi \mathrm{d}x.$$

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Using (13) we may find $\alpha \in \mathrm{R}$ such that for $v \in X^-$,

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$$\geq \alpha. \quad \text{using (13)}$$

By a result of S.J. Li, we deduce $\lim_{\|v\|\to\infty} f_1(v) = +\infty$.

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So φ is coercive. But $\nabla\varphi = 1 - \text{comp}$, so φ satisfies (PS).

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So φ is coercive. But $\nabla \varphi = 1 - \text{comp}$, so φ satisfies (PS).
Using (p_0), φ has a local linking at 0.
By the three critical points theorem, φ has 3 critical points, and f also has 3 critical points. ■

The Pf of Clm 16 is based on

Lem 17. *Let* $\{v_n\} \subset X^-$ *such that* $\nabla f_1(v_n) \to 0$ *and* $\|v_n\| \to \infty$. *Let* $v_n^0 = \|v_n\|^{-1} v_n$. *Then* $\exists v \neq 0$, *up to a subsequence* $v_n^0 \rightharpoonup v$.

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Pf. Assume $v_n^0 \rightharpoonup v$. Then $v_n^0 \to v$ in $L^2(\Omega)$.
Since $|p(x,t)| \leq \Lambda |t|$, $\nabla f_1(v_n) \to 0$, we deduce

$$\begin{aligned}\|v_n\| &\geq \langle \nabla f_1(v_n), v_n \rangle \\ &= \int_\Omega \nabla v_n \cdot \nabla v_n \mathrm{d}x - \int_\Omega p(x, v_n) v_n \mathrm{d}x \\ &\geq \int_\Omega |\nabla v_n|^2 \, \mathrm{d}x - \Lambda \int_\Omega v_n^2 \mathrm{d}x = \|v_n\|^2 - \Lambda |v_n|_2^2 \,. \quad \text{(n large)}\end{aligned}$$

The Pf of Clm 16 is based on

Lem 17. *Let $\{v_n\} \subset X^-$ such that $\nabla f_1(v_n) \to 0$ and $\|v_n\| \to \infty$. Let $v_n^0 = \|v_n\|^{-1} v_n$. Then $\exists v \neq 0$, up to a subsequence $v_n^0 \rightharpoonup v$.*

Pf. Assume $v_n^0 \rightharpoonup v$. Then $v_n^0 \to v$ in $L^2(\Omega)$.
Since $|p(x,t)| \le \Lambda |t|$, $\nabla f_1(v_n) \to 0$, we deduce

$$\begin{aligned}\|v_n\| &\ge \langle \nabla f_1(v_n), v_n \rangle \\ &= \int_\Omega \nabla v_n \cdot \nabla v_n \mathrm{d}x - \int_\Omega p(x, v_n) v_n \mathrm{d}x \\ &\ge \int_\Omega |\nabla v_n|^2 \,\mathrm{d}x - \Lambda \int_\Omega v_n^2 \mathrm{d}x = \|v_n\|^2 - \Lambda |v_n|_2^2 . \text{ (n large)}\end{aligned}$$

So $\|v_n\|^{-1} \ge 1 - \Lambda |v_n^0|_2^2$. Let $n \to \infty$, we get $|v|_2^2 \ge \Lambda^{-1}$. ■

That f_1 satisfies (PS) follows from Lem 17 and (13).

Thank you!

谢 谢！

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